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**DEBT STABILIZATION GAMES IN
THE PRESENCE OF RISK PREMIA**

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Debt Stabilization Games in the Presence of Risk Premia

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Abstract: As a result of the recent financial crisis and the ensuing economic recession, fiscal deficits have soared in many OECD countries. As a consequence, government debt has been on the rise again after a period of stable or declining government debt. In this paper we analyze debt stabilization in a country that features endogenous risk premia, imposed by financial markets that evaluate the probability of debt default by governments. Endogenous risk premia arise by assuming e.g. simple linear relations between risk premia and the level of debt. As a result the real interest rate on government debt can be written as a constant (measuring the risk-free real interest rate corrected for real output growth) plus an endogenous risk premium that depends on the debt level. We bring such endogenous risk premia into the Tabellini (1986) model [22] and analyze the impact of it. This gives rise to a nonlinear differential game. We solve this game for both a cooperative setting and a non-cooperative setting. The non-cooperative game is solved under an open-loop information structure. In particular we present a bifurcation analysis w.r.t. the risk premium parameter.

Keywords: debt stabilization, differential games, nonlinear dynamical systems, economic dynamics.

JEL codes: C7,C62,E6,F4,H6

1 Introduction

In many OECD countries, government debt stabilization has recently moved to the center of interest. This can be explained by the fact that the global financial crisis in 2008 and the ensuing economic slowdown plus fiscal balance deterioration, has shifted government debt levels up significantly and in a rapid manner in many countries. Government debt stabilization is again a high priority on many policymakers' agendas, together with restoring financial stability and economic growth. Financial markets are adding pressures by requiring risk premia on sovereign debt of countries that appear particularly vulnerable to potential default giving their levels of debt and pessimistic prospects for budgetary consolidation and growth recovery. In the euro area, the debt stabilization problems have lead to additional difficulties that are of a more systematic nature: in the absence of national monetary policy instruments, member states are unable to use monetary policy to finance budgetary

deficits neither directly (by "printing money") nor indirectly (by "buying government bonds by the central bank of a member country"). This increases the pressure on other member states to bail-out insolvent member states and the pressure on the ECB to lend to countries that are under threat of financial markets. So far, Ireland, Greece and Portugal have indeed received support programmes to address short-run difficulties in meeting their financial obligations.

Aim of this paper is to analyze government debt stabilization problems in the presence of endogenous risk premia, in a theoretical framework. It is also in this European sovereign debt crisis context that we analyze debt stabilization in a country that faces considerable interest rate differentials with respect to German government bonds and where these differentials have been caused through national idiosyncrasies, i.e. a country that is subject to endogenous risk premia on its government debt. Our analytical framework extends the elegant debt stabilization game of Tabellini [22]. In that stabilization game, the monetary and fiscal authorities are engaged in a dynamic conflict of debt stabilization. Using this model, different types of equilibria can be considered: cooperative ("Pareto") vs. non-cooperative ("Nash") equilibria, hierarchical ("Stackelberg") vs non-hierarchical equilibria, and the implications in terms of adjustment dynamics and steady-states of government debt, budget deficits and money growth be assessed. While the original model is based on a constant interest rate, we introduce a risk premium that depends on the level of debt, changing the original linear debt stabilization game into a non-linear one.

The paper is organized as follows. Section 2 introduces the model we analyze in this paper. In Section 3 we solve the model under a non-cooperative mode of play. The effect of the endogenous risk premium is analyzed in detail. Section 4 considers a cooperative mode of play. Again the effects of the introduction of the endogenous risk premium are studied in detail. Section 5 highlights some additional differences between both modes of play. Finally Section 6 summarizes some conclusions and contains some concluding remarks.

2 Government debt stabilization with endogenous risk premia: a dynamic game

The government debt stabilization model of Tabellini [22] considers the problem of government debt stabilization as a dynamic game between monetary and fiscal authorities. Both are assumed to have an interest in government debt stabilization and their own instrument that contributes to achieve this objective. The fiscal authority can reduce the primary fiscal deficit -either by increasing taxes or by reducing government spending-, reducing thereby the accumulation of debt. The monetary authority can increase monetary financing. Monetary financing takes the form of (base) money growth and in a more modern form of buying government debt by the central bank.

Several extensions of the model exist. Van Aarle et al. [1] consider government debt stabilization issues in a two-country monetary union model extension of the Tabellini model. Weeren et al. [23] and Engwerda [8] analyze the existence and asymptotic behaviour of feedback Nash and open-loop Nash equilibria of LQ dynamic games and the lastmentioned paper applies this analysis to the Tabellini debt stabilization game. Van den Broek [2] situates the debt stabilization game in a moving horizon dynamic game that could e.g. approximate regular elections of new policymakers. It is found that a shorter planning horizon reduces the debt stabilization efforts in the Nash open-loop case. Castren [4] considers the finite time horizon variant of the debt stabilization model to consider electoral effects in the form of end-point debt targets. Di Bartolomeo and Di Gioacchino [5] analyze

the Stackelberg open-loop equilibria of the debt stabilization game, showing that these lead to a further reduction in debt stabilization efforts compared to the Nash case. Figuières [12] considers the Tabellini debt stabilization game in the context of an n-country extension where countries consider to join a monetary union and how this changes the strategic incentives.

The setup of the debt stabilization model is as follows. Consider a country consisting of one fiscal authority and one central bank (CB) that is responsible for the monetary policy. The accumulation of government debt is given by the dynamic government budget constraint that relates government debt, interest payments, monetary financing and primary fiscal deficits:

$$\dot{d}(t) = r(t)d(t) + f(t) - m(t) \quad (1)$$

in which d denotes government debt scaled to the level of national output, r_i denotes the real interest rate (adjusted for the rate of output growth which is assumed to be constant in the remainder of this paper) and f denotes the primary fiscal deficit, also scaled to output. The monetary financing undertaken by the CB, measured as a fraction of aggregate output, is denoted by m .¹ Note also that a negative value of government debt implies that the government has obtained a claim on private sector assets.

The fiscal policymaker sets the primary fiscal deficit with the aim of minimizing its loss function. We assume that the fiscal authority is concerned with deviations of the fiscal deficit and government debt from their target values. The fiscal players' objective is given by

$$L_F = \frac{1}{2} \int_0^\infty e^{-\theta t} \{ (f(t) - \bar{f})^2 + \beta_F (d(t) - \bar{d}_F)^2 \} dt. \quad (2)$$

The CB is concerned about money growth and debt stabilization. As a consequence, the objective function of the CB is given by

$$L_M = \frac{1}{2} \int_0^\infty e^{-\theta t} \{ (m(t) - \bar{m})^2 + \beta_M (d(t) - \bar{d}_M)^2 \} dt. \quad (3)$$

β_F and β_M indicate the relative preferences concerning debt stabilization of the fiscal authorities and the CB, respectively². From the loss functions we see that three factors determine very much the policy actions: (i) the relative weights given to debt stabilization, (ii) the target values for debt, deficit and money and (iii) the initial conditions w.r.t. government debt. Finally, θ denotes the discount factor. We will assume, moreover, that both fiscal and monetary authorities do not allow debt to grow forever³. Their goal is to stabilize debt at some steady state value using policies that also converge to steady states. So, the set of admissible control policies considered in this paper is given by the next set of locally square integrable functions

¹The total money stock $m(t)$ appears in budget constraint (1) conform Tabellini's model and some related papers. More recent papers use the nominal long term interest rate $i(t)$ as the CB's instrument and no money in the government debt accumulation equation (see e.g. [7]). Since we are focusing on Tabellini's model, our approach seems reasonable. To model debt stabilization games with the nominal long term interest rate $i(t)$ as the CB's instrument and no money in the government debt accumulation equation remains a topic for future research.

²Note that, by considering $\bar{f} = \beta_f f(t)$, $\bar{m} = \beta_m m(t)$, $\bar{d}_F = \beta_{df} d(t)$ and $\bar{d}_M = \beta_{dm} d(t)$, this set-up includes the analysis of debt sustainability problems (see e.g. [17] and [16]). We will not further specialize our outcomes for this special case here.

³This assumption of a bounded steady-state level of debt implies the no-Ponzi game condition holds which imposes that the discounted value of debt tends to zero when time goes to infinity (see also the appendix).

$$\mathcal{U} := \{(f(\cdot), m(\cdot)) \in L_{2,loc} \mid \lim_{t \rightarrow \infty} f(t) = f^e, \lim_{t \rightarrow \infty} m(t) = m^e, \lim_{t \rightarrow \infty} d(t) = d^e\}.$$

In the original Tabellini model and the subsequent literature the real interest rate on government debt remained constant so that the accumulation of government debt is determined by a linear first-order differential equation. The recent government debt crises in the EMU countries Portugal, Ireland, Italy, Greece, Spain (PIIGS)⁴ and several other cases, mostly outside the EMU as e.g. Iceland⁵, witness the presence of risk-premia in (sovereign) bond yields in cases where countries have high levels of government debt. In the Greek crisis e.g. the risk premium on Greek bonds vs German bonds reached a level of more than 10 percent.

It seems interesting to bring such endogenous risk premia into the Tabellini model and analyze the effects of it. This is also particularly interesting in relation to the functioning of the monetary union. It has been argued that financial markets could be helpful in disciplining fiscal authorities: countries with high debt would face high risk premia, whereas countries with high fiscal discipline and low debt pay little or no risk premia. However, others have pointed at potential adverse effects on fiscal discipline in monetary unions in case no bail-out clauses are not credible. In that case rising debt in one country could raise doubt that the common central bank and/or the disciplined countries in the end will be forced to bail out the high debt countries. In that case also interest rates in countries with low debt could rise as the result of spillover, contagion from countries with high debt.

Endogenous risk premia arise by assuming e.g. simple linear relations between risk premia and the level of debt. As a result the real interest rate on government debt can be written as a constant (measuring the risk-free real interest rate corrected for real output growth) plus an endogenous risk premium that depends on the debt level. We assume,

$$r(t) = \bar{r} + \alpha d(t). \quad (4)$$

Empirical studies on determinants of risk premia, see e.g. Baldacci and Kumar [6] and the literature therein, confirm indeed that the government debt level is one of the crucial determinants of sovereign bond risk premia. The empirical estimates for α in their literature overview are typically between 0.02 and 0.08. De Grauwe and Ji [13], in an interesting analysis of the current European debt crisis, assume that financial markets initially underpriced risk in the euro area government bond markets. With the advent of the European debt crisis, however, financial markets appear to overreact and overprice these risks in the case of the PIIGS countries. In their empirical estimations, the value of α increases from 0.01 in the pre-crisis period before 2008 to 0.10 after 2008.

⁴Notice that Spain has a (though strongly increasing!) debt-to-GDP ratio well below to that of the other countries mentioned (74,1% in 2011 vis-à-vis 165,1% for Greece, 127,7% for Italy, 112,6% for Ireland and 111,9% for Portugal (see OECD [18]), but is confronted with a huge banking crisis problem.

⁵With a debt-to-GDP ratio of 127,3% in 2011 (see OECD [18]).

3 Non-Cooperative Debt Stabilization: The Nash Open-loop Case

In this section we consider the game (1-4) under the assumption that the players have an open-loop information structure⁶ about the game. We assume that players act non-cooperatively and play a Nash strategy. That is, they look for actions that have the property that a unilateral deviation from these actions makes them worse off. Clearly this type of strategies is just an "approximate" of real-life strategic behavior. However, since punishment strategies might be enforced in case policy makers do not stick to their commitments, the assumption that players will stick to the agreement is not too unrealistic in this setting.

In the Appendix we prove the next theorem.

Theorem 3.1 *If $(f^*(.), m^*(.)) \in \mathcal{U}$ is a set of open-loop Nash strategies for (1-4), there exist a trajectory for debt $d^*(.)$ and an associated costate variable $\mu^*(.)$ that satisfy the set of nonlinear differential equations:*

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{bmatrix} \bar{r} & -1 \\ -\beta_F - \beta_M & \theta - \bar{r} \end{bmatrix} \begin{bmatrix} d(t) \\ \mu(t) \end{bmatrix} + \alpha d(t) \begin{bmatrix} d(t) \\ -2\mu(t) \end{bmatrix} + \begin{bmatrix} \bar{f} - \bar{m} \\ \beta_F \bar{d}_F + \beta_M \bar{d}_M \end{bmatrix}, \quad (5)$$

with $d^*(0) = d_0$ and where both $\lim_{t \rightarrow \infty} d^*(t) = d^e$ and $\lim_{t \rightarrow \infty} \mu^*(t) = \mu^e$ exist. Furthermore the steady state values (d^e, μ^e) satisfy

$$\mu^e := \bar{r}d^e + \alpha d^{e^2} + \bar{f} - \bar{m}, \quad (6)$$

where d^e is a solution of the third order polynomial equation

$$g(d) := -2\alpha^2 d^3 + \alpha(\theta - 3\bar{r})d^2 + \gamma_1 d + \gamma_0 = 0. \quad (7)$$

Here $\gamma_1 := -b - 2\alpha(\bar{f} - \bar{m})$ and $\gamma_0 := \beta_F \bar{d}_F + \beta_M \bar{d}_M + (\bar{f} - \bar{m})(\theta - \bar{r})$ with $b := \bar{r}(\bar{r} - \theta) + \beta_F + \beta_M$. \square

In the proof of the above theorem it is also shown that (5) has no periodic solutions. Furthermore it is shown that if the steady state value of debt is positive, the steady state value of monetary spending is larger than the steady state value of fiscal deficit.

The costate variable μ is the sum of the costate variables associated with the individual optimization problems of the players (see the Appendix). In the open-loop information case, the costate variables do not represent/measure the value at time t of having an additional unit of the state variable at time t to each player (i.e. the shadow price interpretation breaks down in this case see, e.g., [3]). So, $\mu(t)$ can not be directly interpreted as the sum of the additional cost of both players associated with having a one-unit higher debt at time t . However, as we will see later on, the steady state value of μ is also obtained as the steady state of the costate variable associated with the solution of a cooperative equilibrium in which the relative preference concerning debt stabilization of the central bank are changed. Since the costate variable in this cooperative solution does have the shadow price interpretation, it shows that the costate μ in the non-cooperative game is also closely related to the total additional cost of a higher debt level.

⁶That is, at time $t = 0$ both players have all information about the game, determine their actions, which are then enforced as binding agreements for the whole planning horizon.

In this paper we focus on (the steady state value of) debt and the consequences of an endogenous risk premium on it. To calculate the equilibrium debt trajectory and strategies one might proceed as follows. Below we will show that the interesting steady state points of (5) are saddle-points. Using this, one can construct then uniquely from (5) the corresponding saddle paths $(d^*(.), \mu^*(.))$ and therefore the debt trajectory $d^*(.)$. Corollary 6.1 in the Appendix provides the steady state points for the system (25). In case the corresponding linearized system has one unstable and two stable eigenvalues (which is the case in our benchmark Example 3.5) one can calculate from this system (25) (uniquely) the corresponding equilibrium strategies. Another way, which often helps to gain insight into the dynamics of the system, is to consider the first part of solutions of the corresponding long finite planning horizon problem.

Before we analyze the effect of the endogenous risk premium in this model we first recall in the next subsection some results from the literature if this premium is zero.

3.1 No Endogenous Risk Premium

If $\alpha = 0$, the model (1-4) reduces to the Tabellini model [22] (where additionally $\bar{d}_F = \bar{d}_M = 0$ was considered). From [9][Section 7.8.1] we recall the next results concerning a slightly extended version of this Tabellini game.

Proposition 3.2 *Consider the game (1-4) with $\alpha = 0$, $\bar{d}_F = \bar{d}_M$ and $b > 0$.*

1. *This game has a unique set of admissible equilibrium actions that allow for a feedback synthesis.*
2. *This game has a unique set of admissible equilibrium actions if the policymakers are sufficiently impatient, i.e., if $\theta > 2\bar{r}$.*
3. *This game has an infinite number of admissible equilibrium actions if $\theta < 2\bar{r}$. However, all equilibrium actions yield the same closed-loop system. Furthermore, equilibrium actions converge to the same steady state.*
4. *The steady state values of debt, fiscal policy and monetary policy are: $d^e = \frac{\gamma_0}{b}$, $f^e = \bar{f} - \frac{\beta_F(\bar{r}\bar{d}_F + \bar{f} - \bar{m})}{b}$ and $m^e = \bar{m} - \frac{\beta_M}{\beta_F}(f^e - \bar{f})$, respectively.* □

The intuition behind part 3. of this proposition is that in case the uncontrolled growth of debt gets large compared to its discounting in the cost function, there is a large need to stabilize debt since the cost grows exponentially. Therefore many equilibrium strategies occur.

The results in this proposition are (partially) confirmed below. In the Appendix we prove the next result for our slightly more general model.

Proposition 3.3 *Consider the game (1-4) with $\alpha = 0$.*

The game has a unique steady state value of debt $d^e = \frac{\gamma_0}{b}$, for every initial state d_0 , if $b > 0$. If $b < 0$ the game has no open-loop Nash equilibrium unless $d_0 = d^e = \frac{\gamma_0}{b}$. □

So if $b < 0$ the problem has for most initial states no solution, a result also shown by Tabellini. Such cases arise if, e.g., $\theta > r$ and β_F and β_M are relatively small. That is, in cases where the authorities don't care too much about the future development of debt.

3.2 A Phase Plane Analysis with Endogenous Risk Premium

In Figure 1 we plotted, in case $\rho := \frac{(\theta - \bar{r})(\beta_F + \beta_M)}{2\alpha} - (\beta_F \bar{d}_F + \beta_M \bar{d}_M) < 0$, the phase plane diagram for the set of nonlinear differential equations (5) if there is a risk premium (i.e. $\alpha > 0$). The

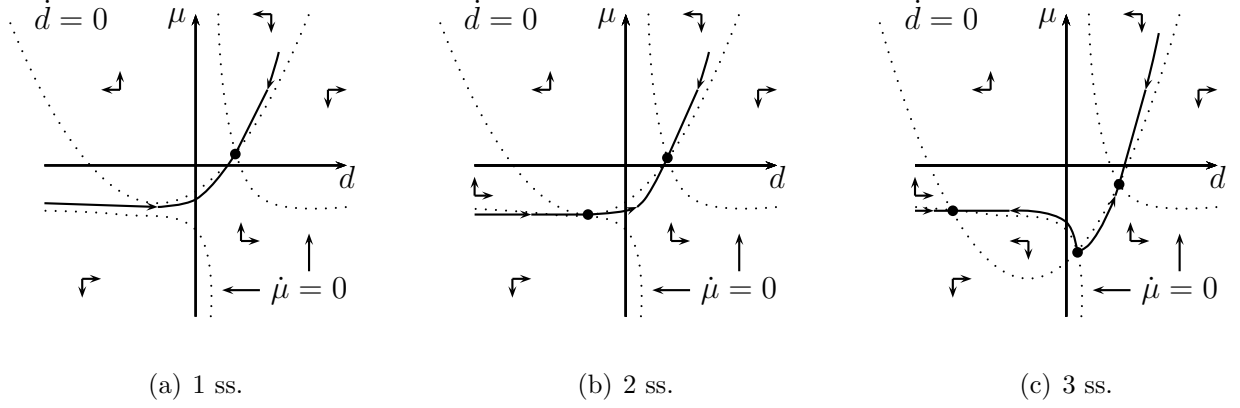


Figure 1: Phase plane diagram of system (5) if $\rho < 0$.

corresponding phase plane diagrams in case $\rho = 0$ and $\rho > 0$ are presented in the Appendix. Figure 1 illustrates that depending on the location of the isoclines three different situations can occur. The phase plane diagram can have either one, two or three steady states. The $\dot{d} = 0$ isocline is given by $\mu = \alpha d^2 + \bar{r}d + \bar{f} - \bar{m}$ whereas the $\dot{\mu} = 0$ isocline equals $\mu = \frac{\theta - \bar{r}}{2\alpha} + \frac{\rho}{\theta - \bar{r} - 2\alpha d}$. Notice that the location of the horizontal asymptote of the $\dot{\mu} = 0$ isocline is $\mu = -\frac{\beta_F + \beta_M}{2\alpha} < 0$. Furthermore the location of the vertical asymptote of the $\dot{\mu} = 0$ isocline, $d = \frac{\theta - \bar{r}}{2\alpha}$, is located to the right of the location of the minimum value of the $\dot{d} = 0$ isocline, i.e. $d = -\frac{\bar{r}}{2\alpha}$. This implies that if $\rho < 0$ and there are three steady states, at least one of them is associated with a negative value of steady state debt, d^e . Only in case $\rho > 0$ it is possible to have 3 positive steady states for the debt. The precise location of the steady states depends on the specific values that the model parameters have. So if, e.g., $4\alpha(\bar{f} - \bar{m}) - r^2 > 0$, it follows that the minimum value of the $\dot{d} = 0$ isocline will be nonnegative and, consequently, the steady state value of μ^e will always be positive. Furthermore, if $\rho < 0$, simulations show that in case there are 3 steady states, for some choice of parameters the middle one will be a node whereas for other choices it will be a focus.

3.3 Number of Steady States with Endogenous Risk Premium

The next proposition, which proof can be found in the Appendix, provides a precise characterization of the number of steady states.

Proposition 3.4 *Let $s := \theta - 3\bar{r}$ and*

$$h(\alpha) := 8\gamma_1^3 - 36\alpha\gamma_0\gamma_1s - 4\alpha\gamma_0s^3 - 108\alpha^2\gamma_0^2 + \gamma_1^2s^2. \quad (8)$$

The game (1-4) has

1. *one steady state value of debt if either i) $h(\alpha) < 0$ or ii) $h(\alpha) = 0$ and $\alpha^2s^2 = 3\gamma_1$;*
2. *two steady state values of debt (from which one only applies as an open-loop equilibrium if $d_0 = d^e$) if $h(\alpha) = 0$ and $\alpha^2s^2 \neq 3\gamma_1$;*

3. three steady state values of debt (from which the middle only applies as an open-loop equilibrium if $d_0 = d^e$) if $h(\alpha) > 0$. \square

The most important difference if one compares both Propositions 3.3 and 3.4 is that the consequence of including a risk premium is that the phase plane diagram always has at least one saddle-point.

Example 3.5 As the benchmark case we will consider in this paper the next parameters: $\theta = 0.1$, $\bar{f} = 0.01$, $\bar{m} = 0$, $\bar{d}_F = 0.6$, $\bar{d}_M = 0.6$, $\beta_F = 0.04$, $\beta_M = 0.04$, $\bar{r} = 0.03$ and $\alpha = 0.05$. Note that this benchmark case essentially emerges for broad -economically relevant- ranges of parameter values (with of course outcomes that will differ in numerical terms). For this case there is a unique steady state $(d^e, \mu^e) = (0.6055, 0.0465)$. It is easily verified that for this benchmark case $h(\alpha) < 0$ for all $\alpha > 0$. So there is a unique steady state for all $\alpha > 0$. That is, independent of the choice of the risk premium we see that the steady state value of debt does not depend on the initial debt. \square

3.4 Effect of Endogenous Risk Premium on Steady State Debt and Actions

Under the assumption that $\frac{\partial g}{\partial d}(\alpha, d^e(\alpha)) \neq 0$ it follows from (7) that d^e is a continuous differentiable function of α and that $d^{e'}(\alpha) = -\frac{\frac{\partial g}{\partial \alpha}}{\frac{\partial g}{\partial d}}$ where the partial derivatives are evaluated at $(\alpha, d^e(\alpha))$. From (28) we have

$$\frac{\partial g}{\partial d} = -6\alpha^2 d^2 + 2\alpha(\theta - 3\bar{r})d + \gamma_1 \text{ and} \quad (9)$$

$$\frac{\partial g}{\partial \alpha} = d^2(\theta - \bar{r} - 2\alpha d) - 2d\mu = d[-4\alpha d^2 + (\theta - 3\bar{r})d - 2(\bar{f} - \bar{m})]. \quad (10)$$

So, e.g., if $\bar{f} \geq \bar{m}$ and $\theta \leq 3\bar{r}$, $\frac{\partial g}{\partial \alpha} < 0$. Consequently, if under these parameter assumptions the game has a unique positive steady state value of debt (and therefore $\frac{\partial g}{\partial d}(\alpha, d^e(\alpha)) < 0$), $d^{e'}(\alpha) < 0$. So the steady state value of debt decreases in those cases if α becomes larger. The intuition is that in case debt grows substantially (compared to the discounting in the cost function again), there is a large need to stabilize the system. The larger α is the larger the destabilization effect becomes. Therefore it is better to reduce debt more substantially in case α gets larger.

In case there is no risk premium, i.e. $\alpha = 0$, $\frac{\partial g}{\partial \alpha} = d[(\theta - 3\bar{r})d - 2(\bar{f} - \bar{m})]$. For our benchmark parameters, with no risk premium, $d^e(0) = 0.6252$. Substitution of this into the previous formula shows that at $\alpha = 0$ for this value of d^e , $\frac{\partial g}{\partial \alpha} < 0$ too. So one can expect that for small values of α for our benchmark parameters $\frac{\partial g}{\partial \alpha} < 0$ too. We verified this numerically in Figure 2. From Figure 2a we observe that the impact of α and hence of the endogenous risk premium on the steady state value of debt is not that high; $\alpha = 0.05$ implies that the steady state value of debt is reduced by approximately 3.3%. Furthermore we observe that the negative relationship between α and the steady state value of debt seems to be almost linear. Moreover it is shown that the steady state risk premium is a (hardly visible in the graph, but slightly) concave function of α which increases to approximately 0.058 for $\alpha = 0.1$. Panel b shows the partial derivatives of g w.r.t. α and d , respectively, at the steady state value as a function of α . In particular this plot indicates that $d^{e'}(\alpha) = -\frac{\frac{\partial g}{\partial \alpha}}{\frac{\partial g}{\partial d}}$ decreases from approximately -0.1 to -0.8 if α increases from 0 to 0.1. So, from this plot we conclude that there is not a strict linear relationship between α and d^e but that the impact of the risk premium on the steady state debt is indeed small.

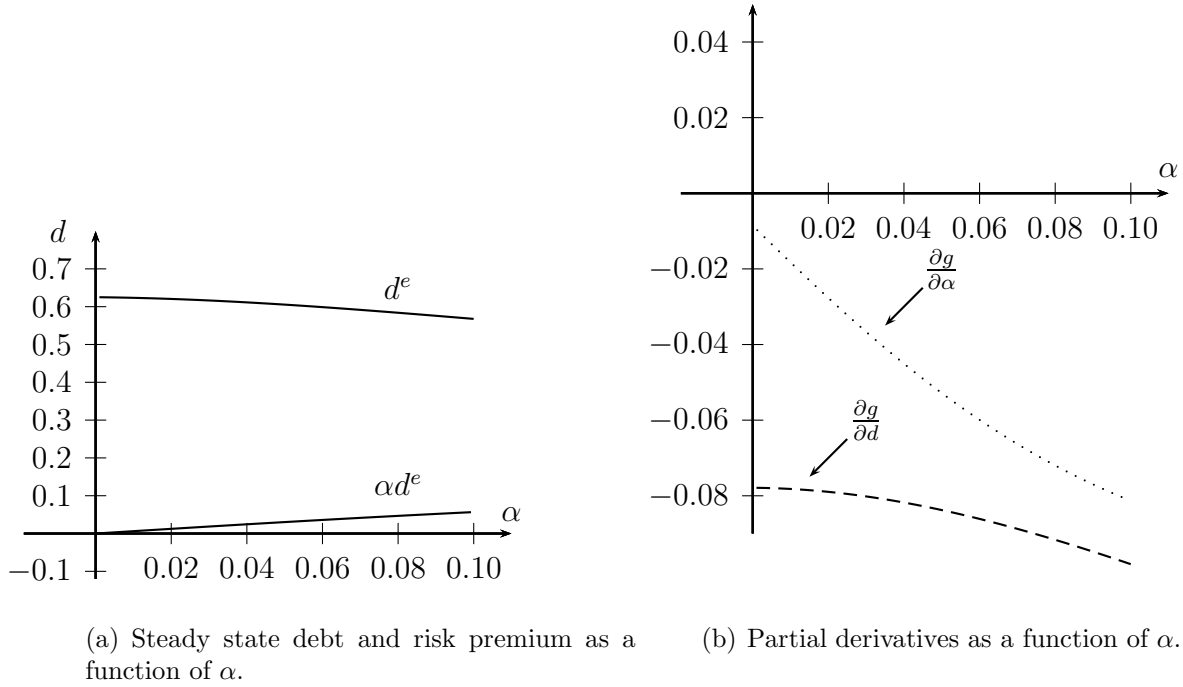


Figure 2: Debt, premium and partial derivatives as a function of α for the benchmark parameters.

In Figure 3 we plotted for the benchmark parameters the corresponding steady state control actions as a function of the risk premium. Since $\frac{\partial f^e}{\partial \alpha} = -\frac{\partial m^e}{\partial \alpha}$, we see a similar behavior of both steady state monetary and fiscal policies. That is, both policies are more actively used if the risk premium increases. We also see that this effect becomes less the larger the risk premium is. Note again that a negative value of the fiscal deficit, f , implies a fiscal surplus.

It is also possible to give an explicit expression for the derivative of the steady state strategies w.r.t. α . For instance $\frac{\partial f^e}{\partial \alpha} = -\frac{\beta_F}{(\theta - \bar{r} - 2\alpha d^e)^2} (d'(\alpha)(\theta - \bar{r} - 2\alpha d^e) + (2d^e + 2\alpha d'(\alpha))(d^e - \bar{d}_F))$, with $d'(\alpha)$ as mentioned above. For the benchmark parameters it follows that $\frac{\partial f^e}{\partial \alpha}(0.05) = -0.1585$ and $\frac{\partial m^e}{\partial \alpha}(0.05) = 0.1585$.

Tabellini proved (in his Proposition 3) that in the non-cooperative case, a reduction in the weight attached to debt stabilization by one player: (i) reduces the adjustment speed, (ii) increases the steady-state stabilization burden for the other player and (iii) increases steady-state debt. Figure 4 illustrates that for small values of α , debt indeed decreases if the weight attached by the policymakers to debt stabilization increases. However, this effect may reverse for larger values of α : in case $\alpha = 0.1$, an increase in the weight attached to debt stabilization, increases steady state debt. In other words, the result found by Tabellini generalizes to the case where the nonlinearity introduced by the risk premium mechanism is not very strong but does no longer hold necessarily when the risk premium mechanism gets stronger (or in other words when financial markets seek to impose stronger compensation for (perceived) solvency risks). The intuition is that if α is large, for smaller values of β_F , due to the large destabilization effect of the risk premium, it is best for the policymaker to make debt as small as possible. However, by increasing β_F the policymaker expresses that he likes

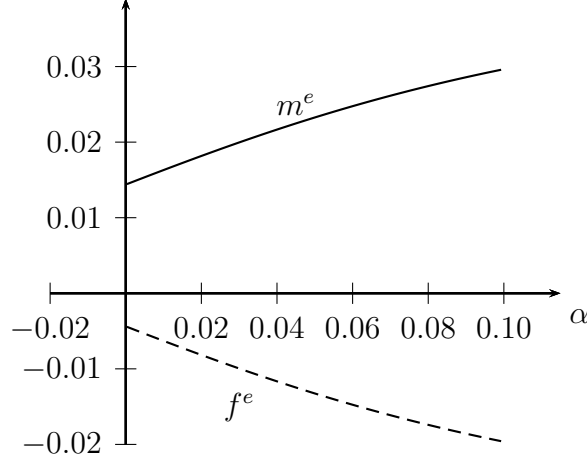


Figure 3: Steady state policies as a function of α for the benchmark parameters.

to control debt to be closer to its target value \bar{d}_F . So optimal steady state debt is forced to move into this direction then.

3.5 A bifurcation analysis w.r.t. α

In this subsection we focus on what the effect of α is in the model.

Recall from Proposition 3.3 that for $\alpha = 0$ the problem has a unique equilibrium point $d^e = \frac{\gamma_0}{b}$ if $b > 0$ and no equilibrium if $b < 0$. Furthermore it is easily verified that $h(\alpha)$ in (8) can be rewritten as

$$h(\alpha) = -64(\bar{f} - \bar{m})^3 \alpha^3 + \delta_2 \alpha^2 + \delta_1 \alpha + b^2(s^2 - 8b), \quad (11)$$

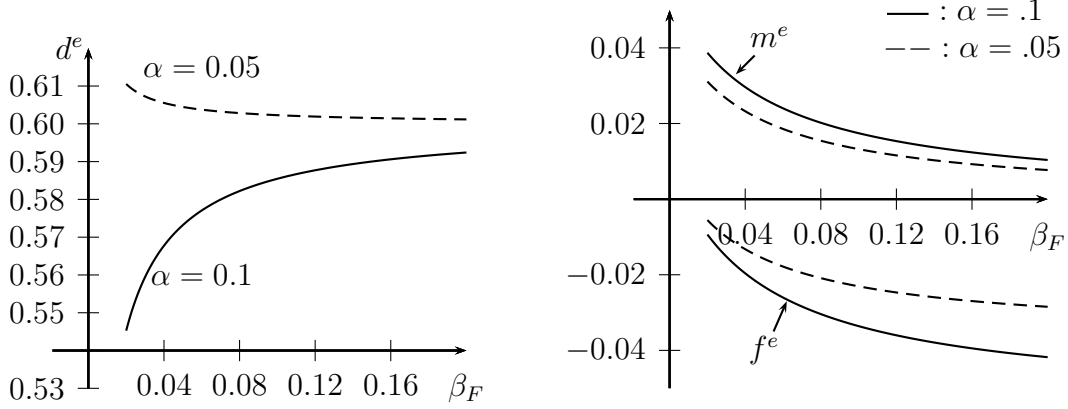
where $\delta_1 := -4(12(\bar{f} - \bar{m})b^2 - (\bar{f} - \bar{m})bs^2 - 9\gamma_0 bs + \gamma_0 s^3)$ and $\delta_2 := -4(-(\bar{f} - \bar{m})^2 s^2 + 24b(\bar{f} - \bar{m})^2 - 18(\bar{f} - \bar{m})\gamma_0 s + 27\gamma_0^2)$. So for small values of α , $h(\alpha) \approx h(0) = b^2(s^2 - 8b)$. Combining the results of both Propositions 3.3 and 3.4 yields then the bifurcation outlined in Table 1.

Table 1: Bifurcation of number of steady states at $\alpha = 0$ and for small values of α .

sign (b)	sign ($s^2 - 8b$)	# ss. if $\alpha = 0$	# ss. if $\alpha > 0$
+	+	1	3
+	-	1	1
-	+	0	3

From the last row of this Table 1 we observe that the introduction of an endogenous risk premium, in cases where no equilibrium exists, implies that three steady states will occur. Furthermore, we see from the first row of this table that in case the premium-free game has an equilibrium two additional steady states may occur if a premium is introduced.

To get an impression how α affects the location of these new steady states, we plotted in Figure 5 steady state debt as a function of α for a set of parameters where $b < 0$ and $h(0) > 0$. So the premium-free game has no equilibrium, whereas for small values of α there are 3 steady state values of debt. From Figure 5a we see that for very small values of α there is one steady state point at $+\infty$, one at $-\infty$ and one at approximately -0.0584 . However, we see that the two saddle-path equilibria



(a) Steady state debt as a function of β_F . (b) Steady state policies as a function of β_F .

Figure 4: Steady state Debt and policies as a function of $\beta_F (= \beta_M)$ for the benchmark parameters.

quickly converge to realistic values of debt. Furthermore, for a value of α slightly larger than 0.14, there is just one steady state value of debt. In case we increase in the simulations \bar{f} to its benchmark again, we see a similar behavior and for $\alpha \approx 0.02$ already there is just one steady state left. So including a relative small risk premium implies that a situation where no stable equilibrium path exists is changed into a realistic situation where such a path does exist.

Next we analyze the number of (positive) steady state values of debt as a function of α for larger values of α .

First some observations concerning the location of the steady states. Notice that, if $\gamma_0 > 0$, for every fixed positive $\bar{\alpha}$ there will be at least one positive steady state d^e since $g(\bar{\alpha}, 0) > 0$, $g(\bar{\alpha}, d) < 0$ for large d and $g(\bar{\alpha}, d)$ is continuous. On the other hand, in case $\gamma_0 < 0$ and $\bar{f} - \bar{m} > 0$, it follows directly from the expression of γ_0 that necessarily $\bar{r} > \theta$. So both $\theta - 3\bar{r}$ and γ_1 are negative too. Using Descartes's rule of signs we have then that for all $\bar{\alpha} > 0$ (7) has only negative roots. So there are no steady state values of debt with a positive sign.

Next consider the case that α is large. From (11) and Proposition 3.4 we immediately deduce that

Corollary 3.6 *Assume α is large. Then, the game (1-4) has*

1. *one steady state value of debt if $\bar{f} > \bar{m}$*
2. *three steady state values of debt if $\bar{f} < \bar{m}$.*

□

Finally we analyze how the number of steady states changes if α increases from zero to infinity. Notice that $h(\alpha)$ is a third degree polynomial in α . Consequently $h(\alpha)$ has at least one and at most three real roots. Therefore the number of steady states might potentially change for positive α from

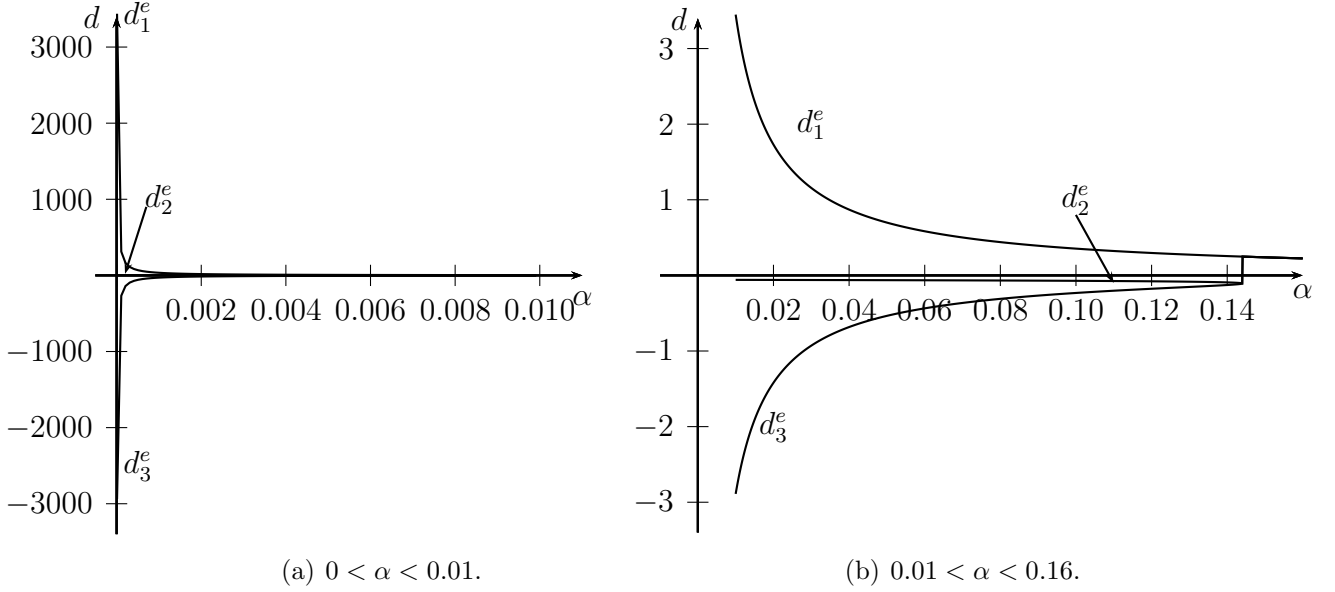


Figure 5: Debt as a function of α with benchmark parameters except $\bar{f} = 0.001$; $\beta_F = \beta_M = 0.00004$.

$3 \rightarrow 1 \rightarrow 3 \rightarrow 1$ in case $\bar{f} > \bar{m}$ and, in case $\bar{f} < \bar{m}$, from $1 \rightarrow 3 \rightarrow 1 \rightarrow 3$ if all roots are positive. But, of course, in case, e.g., $h(\alpha)$ has only negative roots the number of steady states won't change. Example 3.7 illustrates some cases.

Example 3.7

1. $\bar{f} = 0.01$; $\bar{m} = 0$; $\bar{d}_F = 0.6$; $\bar{d}_M = 0.6$; $\beta_F = \beta_M = 0.04$; $\bar{r} = 0.03$; $\theta = 0.75$; yields one steady state for $\alpha \leq 0.12$; 3 steady states for $0.13 \leq \alpha \leq 0.28$ and one steady state again for $\alpha \geq 0.29$.
2. $\bar{f} = 0$; $\bar{m} = 0.01$; $\bar{d}_F = 0.6$; $\bar{d}_M = 0.6$; $\beta_F = \beta_M = 0.0004$; $\bar{r} = 0.03$; $\theta = 0.1$; yields 3 steady states for all $\alpha > 0$.
3. with \bar{m} in 2. replaced by $\bar{m} = 0.05$ we get for $\alpha \leq 0.015$, 3 steady states; for $0.016 \leq \alpha \leq 0.057$, 1 steady state; and for $\alpha \geq 0.058$, 3 steady states again.
4. $\bar{f} = 0$; $\bar{m} = 0.04$; $\bar{d}_F = 0.6$; $\bar{d}_M = 0.6$; $\beta_F = \beta_M = 0.002$; $\bar{r} = 0.03$; $\theta = 0.1$; yields one steady state for $\alpha \leq 0.04$; and 3 steady states for $\alpha \geq 0.05$. Notice that in this case for $\alpha = 0$ we have a negative steady state value of debt which is a saddle point. \square

4 Debt Stabilization: The Cooperative Case

In this section we consider the cooperative case. From [9] we have that Pareto efficient solutions are obtained⁷ by solving for all $\omega \in (0, 1)$ the parameterized optimal control problem

⁷It might be that we still miss some Pareto solutions (see e.g. [19], [20] or [11] for the finite planning horizon case). However a more detailed analysis of this issue is beyond the scope of this paper. We will restrict therefore the analysis to the set of Pareto solutions obtained by minimizing a weighted sum of the cost functions of the players.

$$\min_{f,m} \omega L_F + (1 - \omega) L_M \quad (12)$$

subject to

$$\dot{d}(t) = \bar{r}d(t) + \alpha d^2(t) + f(t) - m(t), \quad d(0) = d_0. \quad (13)$$

Note, that by varying ω between zero and one, one obtains in general a curve of Pareto efficient solutions. So, if the choice of the coordination parameter is not explicitly agreed upon a priori by the players the question arises which solution on the Pareto curve will be selected by the players. We will not further elaborate on this issue here and refer the interested reader to the literature on bargaining theory. In our examples further on, we will focus in most cases on the case that players have equal bargaining strengts, i.e. $\omega = 0.5$ and call the cooperative equilibrium in that case the "social outcome/equilibrium. In the Appendix we prove the next theorem

Theorem 4.1 *If $(f^*(.), m^*(.)) \in \mathcal{U}$ is a set of Pareto efficient strategies for (1-4), there exist a $\omega \in (0, 1)$, a trajectory for debt $d^*(.)$ and an associated costate variable $\mu^*(.)$ that satisfy the set of nonlinear differential equations:*

$$\begin{aligned} \begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} &= \begin{bmatrix} \bar{r} & -\frac{1}{\omega(1-\omega)} \\ -\omega\beta_F - (1-\omega)\beta_M & \theta - \bar{r} \end{bmatrix} \begin{bmatrix} d(t) \\ \mu(t) \end{bmatrix} + \alpha d(t) \begin{bmatrix} d(t) \\ -2\mu(t) \end{bmatrix} \\ &+ \begin{bmatrix} \bar{f} - \bar{m} \\ \omega\beta_F \bar{d}_F + (1-\omega)\beta_M \bar{d}_M \end{bmatrix}, \end{aligned} \quad (14)$$

with $d^*(0) = d_0$ and where both $\lim_{t \rightarrow \infty} d^*(t) = d_c^e$ and $\lim_{t \rightarrow \infty} \mu^*(t) = \mu_c^e$ exist. Furthermore the steady state values (d_c^e, μ_c^e) satisfy

$$\mu_c^e := \omega(1 - \omega)\{\bar{r}d_c^e + \alpha d_c^{e^2} + \bar{f} - \bar{m}\}, \quad (15)$$

where d_c^e is a solution of the third order polynomial equation

$$g_c(d) := -2\omega(1 - \omega)\alpha^2 d^3 + \omega(1 - \omega)\alpha(\theta - 3\bar{r})d^2 + \gamma_1^c d + \gamma_0^c = 0. \quad (16)$$

Here $\gamma_1^c := \omega(1 - \omega)\bar{r}(\theta - \bar{r}) - \omega\beta_F - (1 - \omega)\beta_M - 2\omega(1 - \omega)\alpha(\bar{f} - \bar{m})$ and $\gamma_0^c := \omega\beta_F \bar{d}_F + (1 - \omega)\beta_M \bar{d}_M + \omega(1 - \omega)(\bar{f} - \bar{m})(\theta - \bar{r})$. Moreover,

$$f^*(t) = \bar{f} - \frac{1}{\omega}\mu(t) \text{ and } m^*(t) = \bar{m} + \frac{1}{1 - \omega}\mu(t).$$

□

In the proof of the above theorem it is also shown that (14) has no periodic solutions. Furthermore, similarly as in the non-cooperative case, it is shown that if steady state debt is positive, the steady state value of monetary spending is always larger than the corresponding steady state value of fiscal deficits.

Remark 4.2 In the Appendix we show in Lemma 6.3 that the open-loop equilibrium strategies and cooperative strategies are probably intimately related. Though we do not have a conclusive statement here it is clear that we do have the next result concerning the steady state values. Let $\omega_1 = \sqrt{\omega(1-\omega)}$. Then the steady state values (d^e, μ^e) corresponding to the open-loop equilibrium for the game (1-4) satisfy (6-7) if and only if with β_F replaced by $(1-\omega)\beta_F$ and β_M replaced by $\omega\beta_M$, $(d_c^e, \mu_c^e) := (d^e, \omega_1^2 \mu^e)$ are the steady state values corresponding to the cooperative solution (with weight ω) for this game satisfying (15-16).

This implies, for instance, that the steady state value of debt for the social outcome (i.e. $\omega = 0.5$) is also obtained as the steady state debt of the non-cooperative open-loop game with parameters $\alpha, \theta, \bar{f}, \bar{m}, \bar{r}, 2\beta_F, 2\beta_M, \bar{d}_F$ and \bar{d}_M . That is the steady state debt under a cooperative strategy coincides with that under a non-cooperative equilibrium where the weight attached to the deviation from its target values is doubled by both players. \square

4.1 No Endogenous Risk Premium

If $\alpha = 0$, the optimization problem (12-13) is for every $\omega \in (0, 1)$ a linear quadratic optimal control problem and has a unique solution. After some rewriting we get the next result from the literature (see the Appendix).

Theorem 4.3 *Let $\omega \in (0, 1)$ and $\alpha = 0$. Then the optimization problem (12-13) has the unique solution*

$$[f^*(t) \ m^*(t)]^T = -R^{-1}(B^T K^+ + V)[d^*(t) \ 1]^T,$$

where K^+ is the unique stabilizing solution of (42).

Furthermore, the minimum cost (12) is $[d_0 \ 1]K^+[d_0 \ 1]^T$ and the corresponding minimum costs for the fiscal authority and central bank are $L_i = \frac{1}{2}[d_0 \ 1]\tilde{M}_i[d_0 \ 1]^T$, $i = F, M$, respectively. Here the matrices are as introduced in the proof of the theorem in the Appendix. \square

So, different from the non-cooperative case, for every initial debt and choice of parameters there will always be a unique set of optimal strategies for every fixed choice of the coordination parameter ω .

4.2 Phase Plane Diagrams with Endogenous Risk Premium

The isoclines of system (14) are

$$\begin{aligned} \mu &= \frac{\rho_c}{\theta - \bar{r} - 2\alpha d} - \frac{\omega\beta_F + (1-\omega)\beta_M}{2\alpha} \text{ and} \\ \mu &= \omega(1-\omega)\{\bar{r}d + \alpha d^2 + \bar{f} - \bar{m}\}, \end{aligned}$$

where $\rho_c := \frac{(\theta - \bar{r})(\omega\beta_F + (1-\omega)\beta_M)}{2\alpha} - (\omega\beta_F\bar{d}_F + (1-\omega)\beta_M\bar{d}_M)$. Replacing (see Remark 4.2) β_F by $(1-\omega)\beta_F$ and β_M by $\omega\beta_M$ we get the isoclines multiplied with a factor ω_1^2 we derived for the non-cooperative case. So, basically, we get the same phase plane diagrams.

4.3 Number of Steady States with Endogenous Risk Premium

Notice that with $\tilde{\beta}_F := \frac{\beta_F}{1-\omega}$ and $\tilde{\beta}_M := \frac{\beta_M}{\omega}$, $g_c(d) = \omega_1^2 g(d)$, where in $g(d)$, the parameters β_i are replaced by $\tilde{\beta}_i$, $i = F, M$. Using this we obtain from Proposition 3.4 the next characterization of the number of steady states.

Proposition 4.4 *Let $\omega \in (0, 1)$ be fixed. Furthermore, let $s := \theta - 3\bar{r}$, $\tilde{\gamma}_1^c := \bar{r}(\theta - \bar{r}) - \tilde{\beta}_F - \tilde{\beta}_M - 2\alpha(\bar{f} - \bar{m})$, $\tilde{\gamma}_0^c := \tilde{\beta}_F \bar{d}_F + \tilde{\beta}_M \bar{d}_M + (\bar{f} - \bar{m})(\theta - \bar{r})$ and*

$$h_c(\alpha) := 8\tilde{\gamma}_1^3 - 36\alpha\tilde{\gamma}_0\tilde{\gamma}_1s - 4\alpha\tilde{\gamma}_0s^3 - 108\alpha^2\tilde{\gamma}_0^2 + \tilde{\gamma}_1^2s^2. \quad (17)$$

The game (12-13) has

1. one steady state value of debt if either i) $h_c(\alpha) < 0$ or ii) $h_c(\alpha) = 0$ and $\alpha^2s^2 = 3\tilde{\gamma}_1$;
2. two steady state values of debt (from which one only applies as an equilibrium if $d_0 = d^e$) if $h_c(\alpha) = 0$ and $\alpha^2s^2 \neq 3\tilde{\gamma}_1$;
3. three steady state values of debt (from which the middle only applies as an equilibrium if $d_0 = d^e$) if $h_c(\alpha) > 0$. \square

Combining the results of both Theorem 4.3 and Proposition 4.4 yields then the next bifurcation Table 2 for small values of α , with $\tilde{b} := \bar{r}(\bar{r} - \theta) + \tilde{\beta}_F + \tilde{\beta}_M$.

Table 2: Bifurcation of number of steady states at $\alpha = 0$ for small values of α .

sign ($s^2 - 8\tilde{b}$)	# ss. if $\alpha = 0$	# ss. if $\alpha > 0$
+	1	3
-	1	1

From this table we conclude that in case ω is either approximately zero or approximately one, the introduction of a risk premium will have no effect on the number of steady states. Only for intermediate values of ω the number of steady states might suddenly increase. That is, if in the cooperative case either one of the players has a relative strong impact, the inclusion of a risk premium has not a serious impact on where initial debt will converge to. In case the weight of both players in the cooperative cost function is approximately the same, the inclusion of a risk premium might be substantial. That is the height of initial debt can be important in that case to conclude where the final debt will converge to.

In case the risk premium increases we observe a similar behavior as in the non-cooperative case. In particular Corollary 3.6 applies in this case too. Furthermore a similar behavior concerning the number of steady states can happen as we sketched in the paragraph after Corollary 3.6. Again the number of steady states may switch a couple of times between one and three if the risk parameter α increases.

Example 4.5 Reconsider the benchmark case, Example 3.5, again. Some elementary calculations show that for every $\omega \in (0, 1)$ all coefficients of the third order polynomial $h_c(\alpha)$ are negative. Consequently, $h_c(\alpha) < 0$ for every $\alpha > 0$. So for every coordination parameter the cooperative game has a unique steady state, whatever the value of the risk premium is. \square

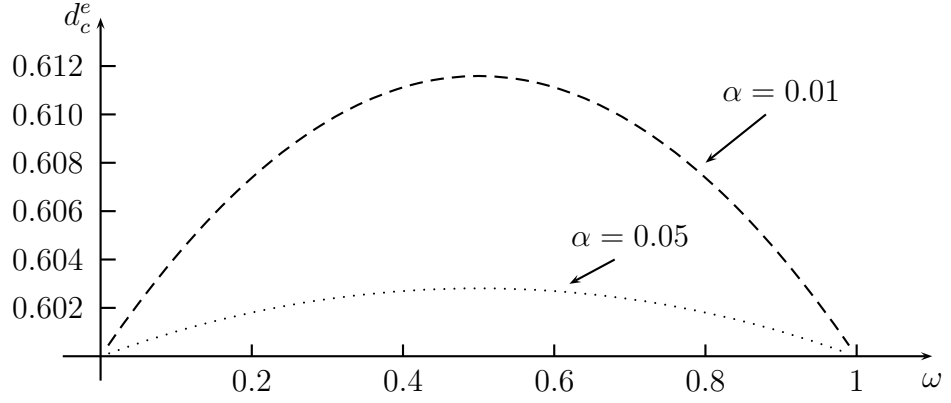


Figure 6: Debt as a function of ω for the benchmark parameters with $\alpha = 0.05$ and $\alpha = 0.01$.

4.4 Effect of Endogenous Risk Premium on Steady State Debt and Actions

In this section we analyze in some more detail the effect of the endogenous risk parameter α on the steady state value of debt d_c^e and actions.

For the benchmark case the steady state value of debt for the social outcome is 0.6028. In Figure 6 we plotted the steady state value of debt as a function of the coordination parameter ω . A fiscal player with a higher (bargaining) weight ω in the cooperative equilibrium, implies that intrinsically a larger share of the debt stabilization burden is shifted to the monetary player resulting in higher money growth and more fiscal consolidation.

We plotted the graph for the benchmark parameters and for the case that the parameter α is changed into $\alpha = 0.01$. The figure illustrates that steady state debt is in this case a decreasing function of α .

The next analysis shows that this property holds for a broad class of parameter values. To that end notice that $\frac{\partial g_c}{\partial \alpha} = \omega_1^2 \frac{\partial g}{\partial \alpha}$. Therefore

$$d_c^{\prime}(\alpha) = -\frac{\partial g_c}{\partial \alpha}(\alpha, d_c^e(\alpha)) / \frac{\partial g_c}{\partial d}(\alpha, d_c^e(\alpha)) = -\omega_1^2 \frac{\partial g}{\partial \alpha}(\alpha, d_c^e(\alpha)) / \frac{\partial g_c}{\partial d}(\alpha, d_c^e(\alpha)).$$

So under the assumption that there is a unique steady state value of debt for the cooperative case, which implies in particular that $\frac{\partial g_c}{\partial d}(\alpha, d_c^e) < 0$, we have that $\text{sign}(d_c^{\prime}(\alpha)) = \text{sign}(\frac{\partial g}{\partial \alpha}(\alpha, d_c^e(\alpha)))$. Consequently, in case e.g. $\bar{f} \geq \bar{m}$, $\theta \leq 3\bar{r}$ and $\gamma_0 \geq 0$ (which implies in particular that $d_c^e \geq 0$), by (10), $d_c^{\prime}(\alpha) < 0$ for all ω . That is, under these parametric conditions, the steady state debt decreases if the endogenous risk premium increases. This, irrespective of the specific choice of the coordination parameter.

In Figure 7a we plotted the steady state values of the control strategies as a function of the coordination parameter. In this figure we also plotted the corresponding non-cooperative steady state policies f^e and m^e . We see that the steady state policies are almost a linear function of the coordination parameter ω . Furthermore we observe from this graph that in case fiscal authorities have a dominant position in determining the cooperative welfare function (i.e. $\omega \approx 1$) there is a change in the sign of fiscal policies compared to the non-cooperative case.

In that case, a steady-state fiscal deficit ($f^e > 0$) is produced accompanied by high money growth; with high values of omega, fiscal and monetary policy are more expansionary than in the non-cooperative case, for small values of omega fiscal and monetary policies are less expansionary in the cooperative case than in the non-cooperative case. On the other hand, if monetary authorities

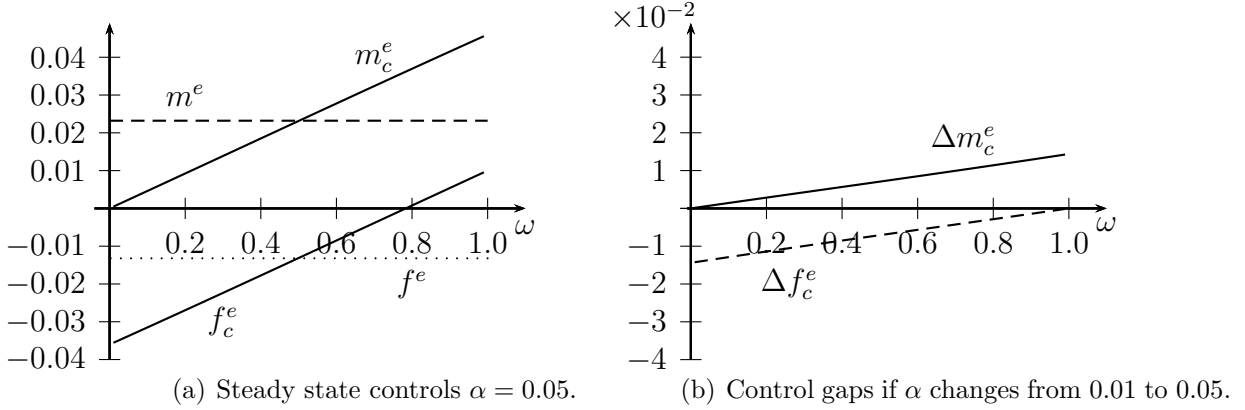


Figure 7: (Change in) steady state controls for benchmark parameters.

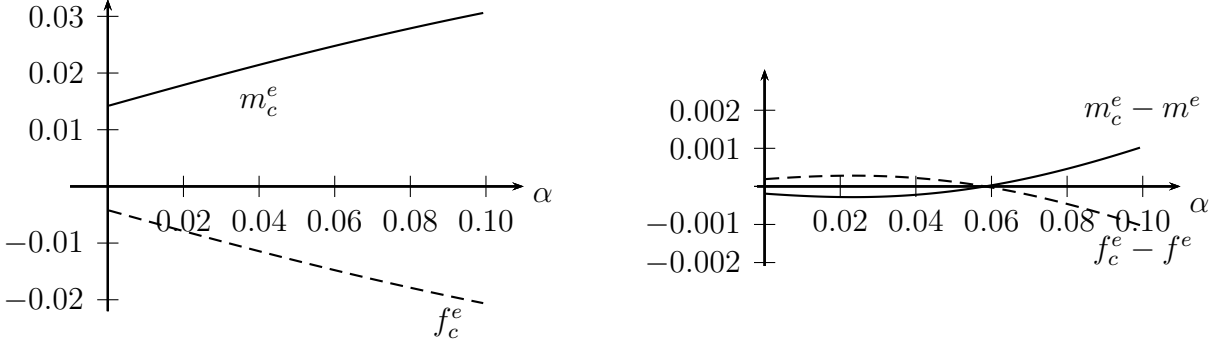
preferences dominate the cooperative welfare function a strategy of almost no monetary policy accompanied by a strong fiscal consolidation is pursued, since the debt stabilization burden is transferred entirely to the fiscal player.

In Figure 7b we plotted the difference, denoted by Δf , between the steady state fiscal strategy at $\alpha = 0.05$ and $\alpha = 0.01$ as a function of the coordination parameter ω . In the same plot we also visualized this difference for the steady state monetary policy. From the graph we see that for $\alpha = 0.05$ a more (negative) active fiscal policy is pursued than for $\alpha = 0.01$. Furthermore we see that for $\omega = 1$ the increase of α has no effect. We observe that this gap is an almost linear function of ω too. Notice that the policy change is, for not too high coordination parameters, relatively quite substantial. For monetary policy we observe an asymmetric reaction compared to fiscal policy. In particular we see that for larger α more (positive) control action is used. This asymmetric reaction is explained by (36) from which we have that independent of the specific choice of α , $\omega f_c^e + (1 - \omega)m_c^e = (1 - \omega)\bar{m} + \omega\bar{f}$. Consequently, $\omega\Delta f_c^e + (1 - \omega)\Delta m_c^e = 0$.

In Figure 8a we plotted for the social outcome (i.e. $\omega = 0.5$) the steady state control actions as a function of the risk premium for the benchmark parameters. Both policies are almost linear functions of α . We see that fiscal policy is a decreasing, slightly convex, function of α and monetary policy an increasing, slightly concave, function of α . Again notice that the relative changes in the level of pursued policies is quite large. In Figure 8b we plotted the difference w.r.t. the corresponding non-cooperative policies. We see that the differences between both modes of play are marginal for both policies. Particularly for small values of the risk premium there is almost no difference between both modes of play. For larger values of the risk premium we see that policies are more actively used in a cooperative mode of play.

5 Some Additional Aspects of the non-cooperative versus the Cooperative Case

In this section we analyze some differences between the non-cooperative and cooperative case in more detail.



(a) Social steady state controls as a function of α .

(b) Control gaps w.r.t. non-cooperative case

Figure 8: Steady state control if $\omega = \frac{1}{2}$ for benchmark parameters.

5.1 Differences in Steady State Variables

First we analyze in some more detail the gap between steady state debt under a cooperative versus a non-cooperative regime. Note that for $\alpha = 0$,

$$d^e(0) - d_c^e(0) = \frac{\beta_F \frac{\omega}{1-\omega} [(\theta - \bar{r})(\bar{r}\bar{d}_F + \bar{f} - \bar{m})] + \beta_M \frac{1-\omega}{\omega} [(\theta - \bar{r})(\bar{r}\bar{d}_M + \bar{f} - \bar{m})] + \beta_F \beta_M \frac{1-2\omega}{(1-\omega)\omega} (\bar{d}_F - \bar{d}_M)}{(\bar{r}(\theta - \bar{r}) - \beta_F - \beta_M)(\bar{r}(\theta - \bar{r}) - \frac{\beta_F}{1-\omega} - \frac{\beta_M}{\omega})}.$$

So, in case $b = (\bar{r}(\bar{r} - \theta) + \beta_F + \beta_M) > 0$, $\theta \geq \bar{r}$, $\bar{f} \geq \bar{m}$ and either $\omega = 0.5$ or $\bar{d}_F = \bar{d}_M$ it is obvious that $d^e(0) - d_c^e(0) > 0$. Assuming additionally that the non-cooperative game will have a unique steady state value of debt for $\alpha > 0$ too, we have by the implicit function theorem again that for small $\alpha > 0$ steady state debt in the non-cooperative case will be larger than steady state debt for the cooperative case too. For the social outcome we plotted this gap as a function of the risk premium parameter α in Figure 9. The plot confirms the above analysis that for small α steady state debt in the non-cooperative case will be larger than in the cooperative case. However, we also see that this gap decreases if α becomes larger and in fact for values of α larger than approximately 0.06, steady state debt in the cooperative case is actually larger than in the non-cooperative case. Interestingly, values between 0.05 and 0.1 are actually typically found in empirical studies that estimate the effect from the debt level on the risk premium. Generally, this leads to the insight that from a debt minimization point of view a cooperative mode of play seems only preferable to a non-cooperative mode of play, in case the strength of the non-linear risk premium mechanism is not too large: beyond a threshold level of the risk premium parameter, α , steady-state debt is actually lower in the non-cooperative equilibrium than in the cooperative equilibrium. This result therefore modifies the original result of Tabellini (1986) who stated in his Proposition 2 that steady-state debt is always lower in the cooperative equilibrium than in the non-cooperative equilibrium (assuming $\alpha = 0$, the linear debt dynamics case).

Finally we notice that the gap between steady state values of monetary and fiscal spending is less in the cooperative case than in the non-cooperative case, provided $d^e \geq d_c^e \geq 0$, as $(m^e - f^e) - (m_c^e - f_c^e) = (r + \alpha(d^e + d_c^e))(d^e - d_c^e) \geq 0$. From Figure 9 we conclude that for the benchmark parameters this

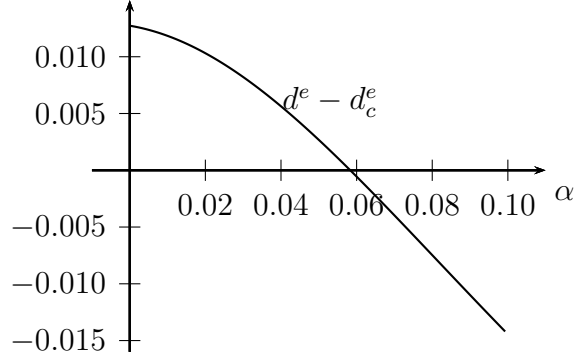


Figure 9: Gap between steady state debt for non-cooperative and social outcome as a function of α for the benchmark parameters.

is the case if the risk premium is not too large (α smaller than 0.06). For a larger impact of this premium the opposite conclusion holds.

5.2 Difference in Closed-loop Behavior

Assuming that there is just one saddle-point we next compare the convergence speed of debt towards this steady state for the non-cooperative and cooperative case, respectively. For the non-cooperative case this is determined by the negative eigenvalue of matrix L in the linearized system (29), i.e. by $|s|$, where

$$s := \frac{\theta - \sqrt{\theta^2 - 4\frac{\partial g}{\partial d}}}{2}. \quad (18)$$

In a similar way we obtain that for the cooperative case the negative eigenvalue of the linearized system (39) around its steady state is

$$s_\omega := \frac{\theta - \sqrt{\theta^2 - 4\frac{1}{\omega_1^2}\frac{\partial g_c}{\partial d}}}{2}.$$

Notice that

$$\begin{aligned} \frac{\partial g}{\partial d} - \frac{1}{\omega_1^2} \frac{\partial g_c}{\partial d} &= -6\alpha^2 d^e + 2\alpha(\theta - 3\bar{r})d^e + \gamma_1 - [-6\alpha^2 d_c^e + 2\alpha(\theta - 3\bar{r})d_c^e + \gamma_1] + \frac{1}{(1-\omega)^2}\beta_F + \frac{1}{\omega^2}\beta_M \\ &= \alpha(d^e - d_c^e)[-6\alpha(d^e + d_c^e) + 2(\theta - 3\bar{r})] + \frac{1}{(1-\omega)^2}\beta_F + \frac{1}{\omega^2}\beta_M. \end{aligned}$$

So in case the game has a positive steady state for $\alpha = 0$ and a unique positive steady state for $\alpha > 0$ too, for small $\alpha > 0$, $\frac{\partial g}{\partial d} - \frac{1}{\omega_1^2} \frac{\partial g_c}{\partial d} > 0$. So in those cases the convergence speed towards its steady state value will be larger in the cooperative case than in the non-cooperative case. That is, debt converges faster to its steady state value in a cooperative setting than in a non-cooperative

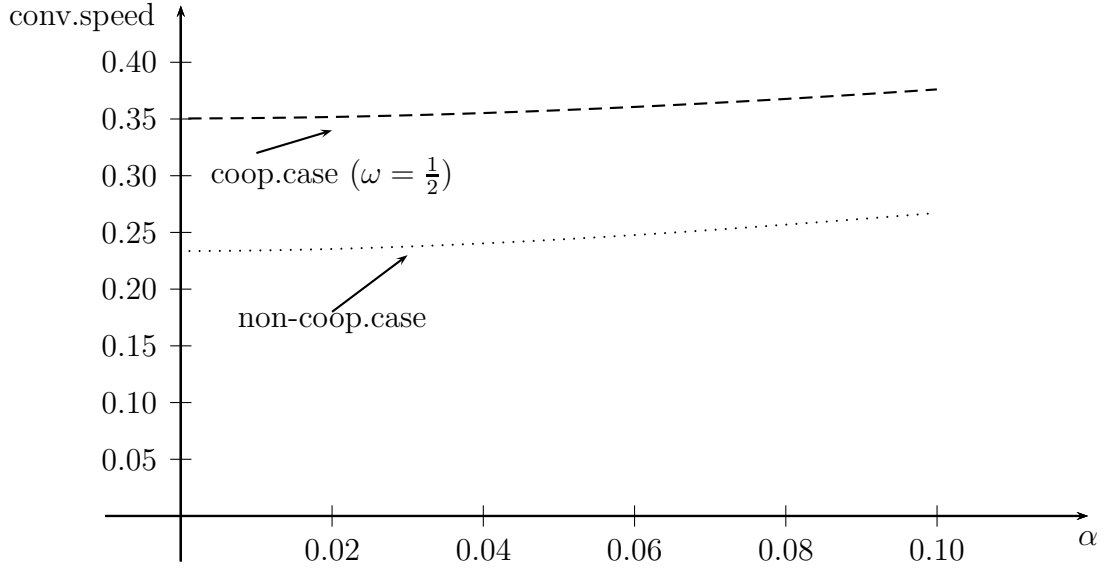


Figure 10: Convergence speed for the benchmark parameters as a function of α for non-cooperative and social outcome.

setting. Notice that this conclusion does not depend on the specific choice of ω . This result confirms the Proposition 1 of Tabellini's paper (1986) that the adjustment speed in a cooperative setting is higher than the adjustment speed in a non-cooperative setting.

Differentiation of (18) w.r.t. α shows, moreover, that $\text{sign}(\frac{\partial s}{\partial \alpha}) = \text{sign}(\frac{\partial \frac{\partial g}{\partial d}}{\partial \alpha}) = \text{sign}(-12\alpha d^e + 2(\theta - 3\bar{r})d^e - 2(\bar{f} - \bar{m}))$. So, if $\theta < 3\bar{r}$, the convergence speed will increase if α increases. That this property also may hold in case $\theta > 3\bar{r}$ is illustrated in Figure 10 for our benchmark parameters. Since $\frac{\partial(\frac{\partial g_c}{\partial \alpha})}{\partial d} = \omega_1^2 \frac{\partial(\frac{\partial g}{\partial \alpha})}{\partial d}$ it follows that a similar conclusion holds w.r.t. the convergence speed in the cooperative case.

Finally notice that an increase of the debt parameter β_F in the cost function usually will lead to a decrease of the convergence speed. This follows directly by differentiation of (18). We have

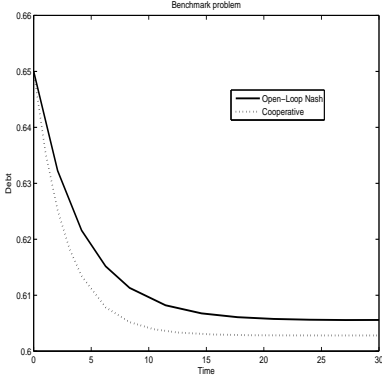
$$s'(\beta_F) = \frac{2}{\sqrt{\theta^2 - 4\frac{\partial g}{\partial d}}} \frac{d(-6\alpha^2 d^{e^2} + 2\alpha(\theta - 3\bar{r})d^e + \gamma_1)}{d\beta_F} = \frac{2}{\sqrt{\theta^2 - 4\frac{\partial g}{\partial d}}} (-1 + \alpha d^{e'}(\beta^F)(-12\alpha d^e + \theta - 3\bar{r})).$$

Therefore, usually, $s'(\beta_F) < 0$. Precise statements can be made by using the implicit function theorem again to calculate $d^{e'}(\beta^F)$. This is, however, left as an exercise to the reader here.

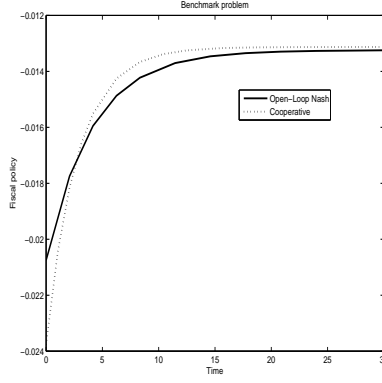
5.3 Adjustment Dynamics

It is interesting to analyze the adjustment dynamics of debt, deficits and money growth in our benchmark case, Example 3.5, introduced earlier. It allows to illustrate the analytical results w.r.t. long-run properties derived in Section 3 and 4 and to give the reader an insight into the typical adjustment dynamics produced by the dynamic debt stabilization game. Figures 11-13⁸ provide the

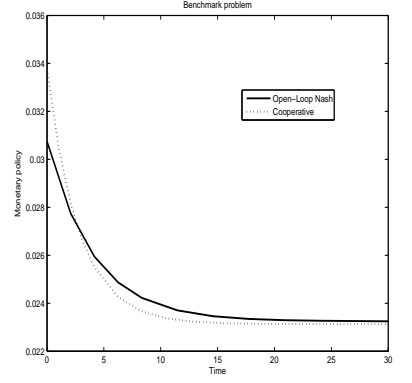
⁸These figures visualize the first part of the time plots generated by considering a long planning horizon (instead of infinity). The last parts of these plots are not shown because they show behavior that is due to the consideration of a finite planning horizon.



(a) debt

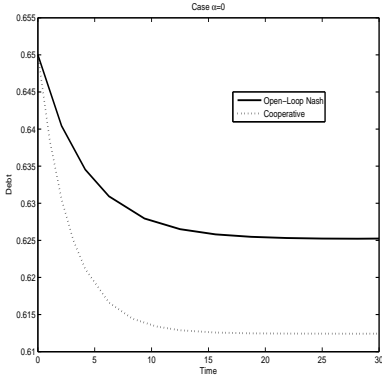


(b) fiscal policy

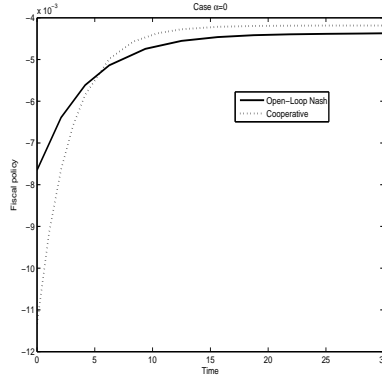


(c) monetary policy

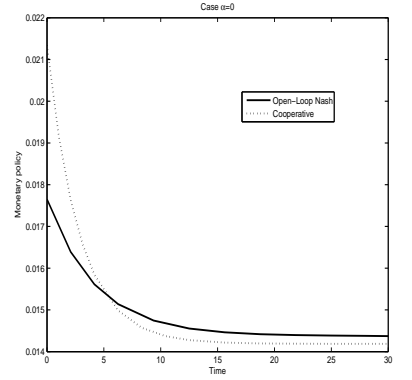
Figure 11: Dynamic adjustment for $\alpha = 0.05$ with benchmark parameters.



(a) debt



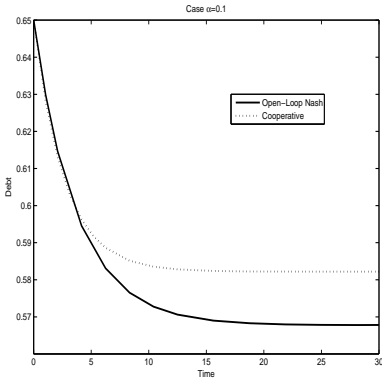
(b) fiscal policy



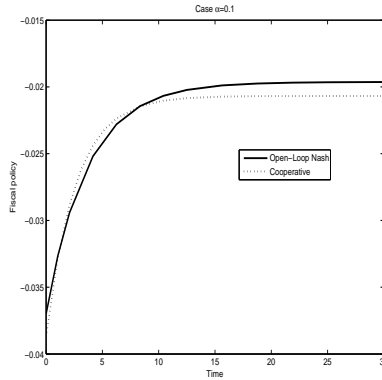
(c) monetary policy

Figure 12: Dynamic adjustment for $\alpha = 0$ with benchmark parameters.

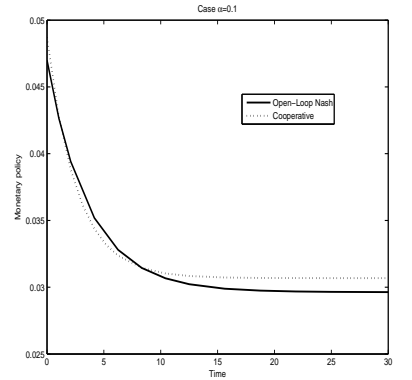
adjustment dynamics produced by this example. In Figure 11 a value of $\alpha = 0.05$ is used. For comparison, we also provide outcomes in case $\alpha = 0$ (Figure 12) and $\alpha = 0.1$ (Figure 13).



(a) debt



(b) fiscal policy



(c) monetary policy

Figure 13: Dynamic adjustment for $\alpha = 0.1$ with benchmark parameters.

In case of the relatively modest risk premium mechanism, $\alpha = 0.05$, we observe that debt adjusts

faster under cooperation than in the non-cooperative case and steady-state debt is higher in the non-cooperative case. In the cooperative case, both policymakers act initially more decisive on debt stabilization, inducing higher money growth and higher fiscal surpluses than in the non-cooperative case. This enables a lower effort in the long run: in the long run money growth and fiscal surplus are actually smaller under cooperation than under non-cooperation. If we compare this with the case of no risk premium, $\alpha = 0$ (Figure 12), we arrive at a similar picture: debt is lower in the long-run under cooperation, money growth is initially higher and fiscal consolidation stronger under cooperation than in the absence of cooperation but in the long run the opposite holds. Moreover, we see that with $\alpha = 0$, the difference between long-run debt in the cooperative and non-cooperative case increases compared to the case with $\alpha = 0.05$: the endogenous risk-premium is able -in this case- to reduce the distance between the cooperative and non-cooperative case by incorporating the externality effects in the debt stabilization game.

Outcomes are fundamentally changed if we introduce a stronger risk premium mechanism, $\alpha = 0.1$ (Figure 13). In that case, debt is actually stabilized stronger in case of non-cooperation. In steady-state, debt, the fiscal surplus and money growth are larger under cooperation than under non-cooperation. From Figure 9 we anticipated already such a result that would start to appear around $\alpha > 0.06$. A strong endogenous risk premium, in other words, changes quite substantially the optimal strategies and the resulting debt stabilization over time. Cooperation is no longer the preferred equilibrium as welfare losses under cooperation now exceed losses under non-cooperation. These effects can be understood if one realizes that the adjustment dynamics and long-run equilibrium of the debt stabilization game are fundamentally changed by the presence of financial markets that impose an endogenous risk premium. As already discussed before, the results obtained by Tabellini on the linear case without risk premia, therefore, do not generally hold in the presence of an endogenous risk premium (they do essentially in case the risk premium mechanism is small e.g. in times of very tranquil financial markets, but once the strength of the endogenous risk premium increases, the debt stabilization game and its outcomes changes fundamentally). Note also that the long-run differences between the cooperative and non-cooperative equilibrium increase also when we compare $\alpha = 0.05$ and $\alpha = 0.1$, as was also observed when we compared $\alpha = 0.05$ and $\alpha = 0$ (and anticipated from Figure 8 and 9).

Comparing all three cases, we observe that in this benchmark case both under cooperation and non-cooperation, steady-state debt is lower in the presence of a stronger endogenous risk premium mechanism. A stronger endogenous risk premium, increases the long-run interest rate burden from debt and therefore a higher long-run stabilization need: steady-state money growth and fiscal surpluses are larger when the endogenous risk premium mechanism is stronger.

6 Concluding Remarks

In this paper we analyzed the impact of an endogenous risk premium on debt and policies in a simple dynamic game between the fiscal and monetary authorities in a country. This paper extends the Tabellini (1986) model by the inclusion of a debt-dependent risk premium. Both a cooperative mode and a non-cooperative mode of play are considered.

Due to its simple structure it is possible to solve (at least partially) the model analytically. In fact we could completely solve analytically the cooperative game. For the non-cooperative case we could derive analytic expressions for the evolution of debt by a coordinate transformation. This enhanced to analyze debt from the resulting necessary conditions in a two-dimensional system. Unfortunately,

it is not possible to track the corresponding non-cooperative policies analytically. However, we indicated how in some cases (including our benchmark case that represents outcomes under a broad range of -economically relevant- parameter values) one can still compute these policies numerically.

We showed that by including this risk premium the game always has at least one and at most three equilibria. The number of equilibria may fluctuate with the value of the risk premium. In case there is one equilibrium we showed that for every initial debt, debt converges to this stationary value. In case three different equilibria occur, it depends on initial debt which equilibrium realizes. The middle equilibrium is unstable and the outer two equilibria are stable. The case that two equilibria occur happens only in rare cases and an in depth analysis of this case is therefore skipped.

We also observed that every cooperative equilibrium can be realized as a non-cooperative equilibrium by changing the relative preference concerning debt stabilization of the central bank with a factor $1/\omega$ and that of the fiscal authority with a factor $1/(1 - \omega)$, where ω is the value of the coordination parameter used in the cooperative welfare function.

For the benchmark case we analyzed the model in some more detail. Some conclusions from this study are the following. The parameter that measures the strength of the risk premium mechanism, is of crucial importance in the debt stabilization game. Under both modes of play equilibrium debt decreases in case the strength of the risk premium parameter increases. Nevertheless, the effect of the risk premium on the level of equilibrium debt seems to be not as strong as the effect on pursued policies (the instruments of the players).

From a debt minimization point of view a cooperative mode of play seems only preferable to a non-cooperative mode of play in case the strength of the non-linear risk premium mechanism is not too large: beyond a threshold level of the risk premium parameter, steady-state debt is actually lower in the non-cooperative equilibrium than in the cooperative equilibrium. This result therefore modifies the original result of Tabellini (1986) that steady-state debt is always lower in the cooperative equilibrium than in the non-cooperative equilibrium. Another result that generalizes only partially to the debt stabilization game with an endogenous risk premium, is the finding by Tabellini that in the non-cooperative case, a reduction of the weight attached to debt stabilization always increases steady-state debt, increases the adjustment burden of the other player and reduces the adjustment speed. If the nonlinearity is small, this result is also likely to be present, but in case the non-linearity is strong, we showed that the first-mentioned result is no longer necessarily present as the strong nonlinearity totally changes the dynamic game and the optimal strategies.

Furthermore, we found that the convergence speed towards the equilibrium is larger under a (social) cooperative mode of play than under a non-cooperative mode of play. The introduction of the endogenous risk premium does not affect this result found by Tabellini.

Finally we observed that in a cooperative mode of play different equilibria may occur only in case the preferences of both players are approximately treated on an equal footing. Moreover we see that from a debt minimization point of view the social outcome gives the highest steady-state debt.

From our general analysis it follows that the results obtained for the benchmark case may change significantly if different parameter settings occur. As already mentioned, for instance, multiple equilibria can occur. One topic for future research could be to identify parameter settings yielding similar behavior.

Another point of interest is to analyze this problem under the assumption that the policy makers use linear state feedback policy rules. Since the optimal feedback parameters become in that case a function of the initial debt, this brings on an additional dimension to the problem.

Finally, last but not least, it is interesting to adapt the model for the nominal long term interest

rate and to extend this model to a multi-county model and analyze, e.g., the dispute about the bail-out clause we mentioned in the introduction section of this model.

Appendix

The Open Loop Case

Proof of Theorem 3.1:

Let $(f^*(.), m^*(.)) \in \mathcal{U}$ be a set of open-loop Nash strategies and $d^*(.)$ the corresponding debt trajectory.

The Hamiltonians for the fiscal and monetary player for (1-3) with endogenous risk premium (4) are

$$H_F := \frac{1}{2}e^{-\theta t}(f - \bar{f})^2 + \frac{1}{2}e^{-\theta t}\beta_F(d - \bar{d}_F)^2 + \lambda_F(\bar{r}d + \alpha d^2 + f - m) \quad (19)$$

and

$$H_M := \frac{1}{2}e^{-\theta t}(m - \bar{m})^2 + \frac{1}{2}e^{-\theta t}\beta_M(d - \bar{d}_M)^2 + \lambda_M(\bar{r}d + \alpha d^2 + f - m), \quad (20)$$

respectively. It is easily verified that this is a normal problem and by Pontryagin's maximum principle there exist continuous and piecewise continuous differentiable functions $\lambda_i^*(.)$, $i = F, M$, $f^*(.), m^*(.)$ and $d^*(.)$ that satisfy the equations:

$$\dot{d}(t) = \bar{r}d(t) + \alpha d^2(t) + f(t) - m(t), \quad d(0) = d_0, \quad (21)$$

$$-\dot{\lambda}_F = \frac{\partial H_F}{\partial d} = e^{-\theta t}\beta_F(d - \bar{d}_F) + \lambda_F(\bar{r} + 2\alpha d), \quad (22)$$

$$-\dot{\lambda}_M = \frac{\partial H_M}{\partial d} = e^{-\theta t}\beta_M(d - \bar{d}_M) + \lambda_M(\bar{r} + 2\alpha d), \quad (23)$$

where $f^*(t) = \bar{f} - e^{\theta t}\lambda_F^*(t)$ and $m^*(t) = \bar{m} + e^{\theta t}\lambda_M^*(t)$ (since $\frac{\partial H_F}{\partial f} = \frac{\partial H_M}{\partial m} = 0$ and both $\frac{\partial^2 H_F}{(\partial f)^2} > 0$ and $\frac{\partial^2 H_M}{(\partial m)^2} > 0$).). Since by assumption $f^*(t)$ and $m^*(t)$ converge, it follows that $\lambda_i^*(t)$, $i = F, M$, converge exponentially to zero. This implies in particular that the limiting transversality conditions $\lim_{t \rightarrow \infty} \lambda_i^*(t)(d(t) - d^*(t)) \geq 0$ hold. Consequently, the above necessary conditions are sufficient too in case for instance both minimized Hamiltonians are convex in d along the optimal shadow price paths. Some elementary calculations show that this is the case iff $\beta_F + 2\alpha(\bar{f} - f^*(t)) \geq 0$ and $\beta_M + 2\alpha(\bar{m} - m^*(t)) \geq 0$, respectively.

Substitution of f and m into the above equations shows that $f^*(.), m^*(.)$ and $d^*(.)$ solve the set of nonlinear differential equations:

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\lambda}_F(t) \\ \dot{\lambda}_M(t) \end{bmatrix} = \begin{bmatrix} \bar{r} & -e^{\theta t} & -e^{\theta t} \\ -\beta_F e^{-\theta t} & -\bar{r} & 0 \\ -\beta_M e^{-\theta t} & 0 & -\bar{r} \end{bmatrix} \begin{bmatrix} d(t) \\ \lambda_F(t) \\ \lambda_M(t) \end{bmatrix} + \alpha d(t) \begin{bmatrix} d(t) \\ -2\lambda_F(t) \\ -2\lambda_M(t) \end{bmatrix} + \begin{bmatrix} \bar{f} - \bar{m} \\ \beta_F \bar{d}_F e^{-\theta t} \\ \beta_M \bar{d}_M e^{-\theta t} \end{bmatrix}. \quad (24)$$

Or, introducing $\mu_i := e^{\theta t}\lambda_i$, $i = F, M$,

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}_F(t) \\ \dot{\mu}_M(t) \end{bmatrix} = \begin{bmatrix} \bar{r} & -1 & -1 \\ -\beta_F & \theta - \bar{r} & 0 \\ -\beta_M & 0 & \theta - \bar{r} \end{bmatrix} \begin{bmatrix} d(t) \\ \mu_F(t) \\ \mu_M(t) \end{bmatrix} + \alpha d(t) \begin{bmatrix} d(t) \\ -2\mu_F(t) \\ -2\mu_M(t) \end{bmatrix} + \begin{bmatrix} \bar{f} - \bar{m} \\ \beta_F \bar{d}_F \\ \beta_M \bar{d}_M \end{bmatrix}. \quad (25)$$

Notice that, since by assumption $f^*(.)$ and $m^*(.)$ converge, $\mu_i^*(t)$, $i = F, M$, converges too and

$$f^e = \bar{f} - \mu_F^e \text{ and } m^e = \bar{m} + \mu_M^e, \text{ respectively.} \quad (26)$$

Consequently, with $\mu(t) := \mu_F(t) + \mu_M(t)$, $\lim_{t \rightarrow \infty} \mu^*(t)$ also exists. So both $\frac{d\{d^*(t)\}}{dt}$ and $\frac{d\{\mu^*(t)\}}{dt}$ converge to zero. Using this in (25) it follows, by adding the second and third equation, that debt and $\mu(t)$ solve the differential equation⁹:

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{bmatrix} \alpha d^2(t) - \mu(t) + \bar{r}d(t) + \bar{f} - \bar{m} \\ (\theta - \bar{r} - 2\alpha d(t))\mu(t) + \beta_F(\bar{d}_F - d(t)) + \beta_M(\bar{d}_M - d(t)) \end{bmatrix} =: \begin{bmatrix} f_1(d, \mu) \\ f_2(d, \mu) \end{bmatrix}. \quad (27)$$

Notice that $\frac{\partial f_1}{\partial d} + \frac{\partial f_2}{\partial \mu} = \theta$. So, if $\theta \neq 0$, by Bendixson's theorem (see e.g. [9][p.88]), this system of differential equations has no periodic solutions. Furthermore the steady state values are obtained as the solutions d^e of

$$g(d) := (\alpha d^2 + \bar{r}d + \bar{f} - \bar{m})(\theta - \bar{r} - 2\alpha d) + \beta_F(\bar{d}_F - d) + \beta_M(\bar{d}_M - d) = 0, \quad (28)$$

with $\mu^e := \bar{r}d^e + \alpha d^{e^2} + \bar{f} - \bar{m}$. Some rewriting of (28) shows that d^e is the solution of the third order polynomial equation (7). Moreover notice that (27) can be rewritten as (5).

Finally notice that, if $d^e > 0$, $m^e - f^e = \bar{m} - \bar{f} + \mu^e = \bar{r}d^e + \alpha d^{e^2} \geq 0$. \square

Corollary 6.1 *From (25) it follows that the steady state values are $\mu_i^e = \frac{\beta_i(d^e - \bar{d}_i)}{\theta - \bar{r} - 2\alpha d^e}$, $i = F, M$. So, by (26), the corresponding equilibrium actions converge to the steady state values $f^e := \bar{f} - \mu_F^e$ and $m^e := \bar{m} + \mu_M^e$, respectively.*

If we assume additionally $\beta_M = \beta_F$ and $\bar{d}_F = \bar{d}_M$ we get $\mu_F^e = \mu_M^e$. Furthermore we have then that $f^e + m^e = \bar{f} + \bar{m}$ and, consequently, $\frac{\partial f^e}{\partial \alpha} = -\frac{\partial m^e}{\partial \alpha}$ and $\frac{\partial f^e}{\partial \theta} = -\frac{\partial m^e}{\partial \theta}$. \square

Proof of Proposition 3.3.

First notice that, since $\alpha = 0$, the conditions of Theorem 3.1 are sufficient too (see the proof of Theorem 3.1).

From (5) it follows that if $\alpha = 0$ the steady state is $\begin{bmatrix} d^e \\ \mu^e \end{bmatrix} = \frac{1}{b} \begin{bmatrix} \theta - \bar{r} & 1 \\ \beta_F + \beta_M & \bar{r} \end{bmatrix} \begin{bmatrix} \bar{f} - \bar{m} \\ \beta_F \bar{d}_F + \beta_M \bar{d}_M \end{bmatrix}$.

Now, $\begin{bmatrix} \bar{r} & -1 \\ -\beta_F - \beta_M & \theta - \bar{r} \end{bmatrix} = SDS^{-1}$, with

$$S = \begin{bmatrix} \frac{\theta - 2\bar{r} + \sqrt{\theta^2 + 4b}}{2(\beta_F + \beta_M)} & \frac{\theta - 2\bar{r} - \sqrt{\theta^2 + 4b}}{2(\beta_F + \beta_M)} \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} \frac{\theta - \sqrt{\theta^2 + 4b}}{2} & 0 \\ 0 & \frac{\theta + \sqrt{\theta^2 + 4b}}{2} \end{bmatrix}.$$

So the eigenvalues of this matrix are $\frac{\theta \pm \sqrt{\theta^2 + 4b}}{2}$. Consequently, the steady state is a saddle-point if $b > 0$. Therefore, choosing $\mu(0)$ such that $(d(0), \mu(0))$ is on the saddle-path of (5) (which is the line corresponding with the first column of matrix S in this case (and from which it is easily verified that it is not vertical)), yields an appropriate solution.

Notice that if $b < 0$, the steady state will be an unstable node. So, if $b < 0$, the problem has no solution if $d_0 \neq d^e$ since all candidate optimal trajectories diverge from the steady state. If $d_0 = d^e$,

⁹By introducing another variable $\nu(t) := \mu_F(t) - \mu_M(t)$ we have that $(d(t), \mu_F(t), \mu_M(t))$ solve (25) if and only if $(d(t), \mu(t), \nu(t))$ solve (27) and the differential equation $\dot{\nu}(t) = (\theta - \bar{r} - 2\alpha d(t))\nu(t) + \beta_F(\bar{d}_F - d(t)) - \beta_M(\bar{d}_M - d(t)) =: f_3(d(t), \nu(t))$. So by considering this coordination transformation we triangularized the system dynamics.

the stationary trajectory satisfies the conditions of Theorem 3.1. So only for this initial condition we have an open-loop Nash equilibrium.

If $b > 0$ it follows directly from (25) that the steady state value of debt equals $d^e = \frac{\gamma_0}{b}$, $\mu_F^e = \frac{\beta_F(\bar{d}_F b - \gamma_0)}{(\bar{r} - \theta)b}$ and $\mu_M^e = \frac{\beta_M(\bar{d}_M b - \gamma_0)}{(\bar{r} - \theta)b}$. So that $\beta_M \mu_F^e - \beta_F \mu_M^e = \frac{\beta_F \beta_M (\bar{d}_F - \bar{d}_M)}{\bar{r} - \theta}$. By substituting in these expressions $\bar{d}_F = \bar{d}_M$ and using (26) one obtains the results advertised in part 4. of Proposition 3.2. \square

Proof of Proposition 3.4.

In case there is a steady state, from (5), the linearized system around this steady state is described by:

$$\dot{y} = Ly := \begin{bmatrix} \bar{r} + 2\alpha d^e & -1 \\ -\beta_F - \beta_M - 2\alpha \mu^e & \theta - \bar{r} - 2\alpha d^e \end{bmatrix} y. \quad (29)$$

By straightforward calculations one can verify that the eigenvalues of matrix L are $\frac{\theta \pm \sqrt{\theta^2 - 4\frac{\partial g}{\partial d}}}{2}$. So (ignoring the special case that one eigenvalue becomes zero) we see that the steady states will be either unstable or saddle-points. From the sign scheme of the partial derivative $\frac{\partial g}{\partial d}$ at the steady state(s) we can directly deduce the behavior near the steady states. We visualized the results in Table 1B.

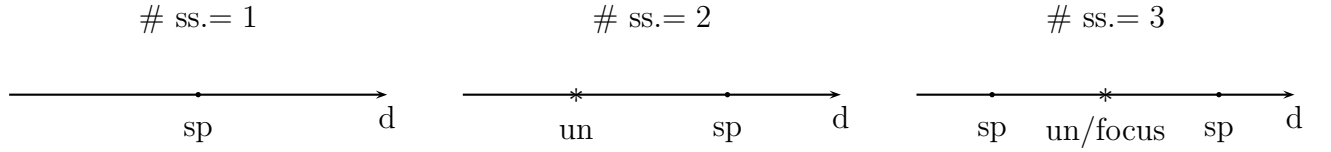


Table 1B: sp=saddle point; un=unstable node

Notice that the unstable steady state will be a focus if $\theta^2 - 4\frac{\partial g}{\partial d} < 0$ and a node if $\theta^2 - 4\frac{\partial g}{\partial d} > 0$.

To determine the conditions under which (7) has either one, two or three solutions let $\gamma_3 := -2\alpha^2$ and $\gamma_2 := \alpha(\theta - 3\bar{r})$. With this notation (7) is a cubic equation of the form

$$\gamma_3 d^3 + \gamma_2 d^2 + \gamma_1 d + \gamma_0 = 0. \quad (30)$$

Next consider the discriminant

$$\Delta := 18\gamma_3\gamma_2\gamma_1\gamma_0 - 4\gamma_2^3\gamma_0 + \gamma_2^2\gamma_1^2 - 4\gamma_3\gamma_1^3 - 27\gamma_3^2\gamma_0^2. \quad (31)$$

Then (see e.g. [14]) (30) has one real root if $\Delta < 0$; a multiple real root if $\Delta = 0$; and three distinct real roots if $\Delta > 0$.

Simple calculations show that in case there is a multiple root this root occurs three times if and only if $\gamma_2^2 = 3\gamma_1$. With $s := \theta - 3\bar{r}$ one verifies that

$$\Delta = \alpha^2(8\gamma_1^3 - 36\alpha\gamma_1\gamma_0s - 4\alpha\gamma_0s^3 - 108\alpha^2\gamma_0^2 + \gamma_1^2s^2).$$

From this the advertised result is clear. \square

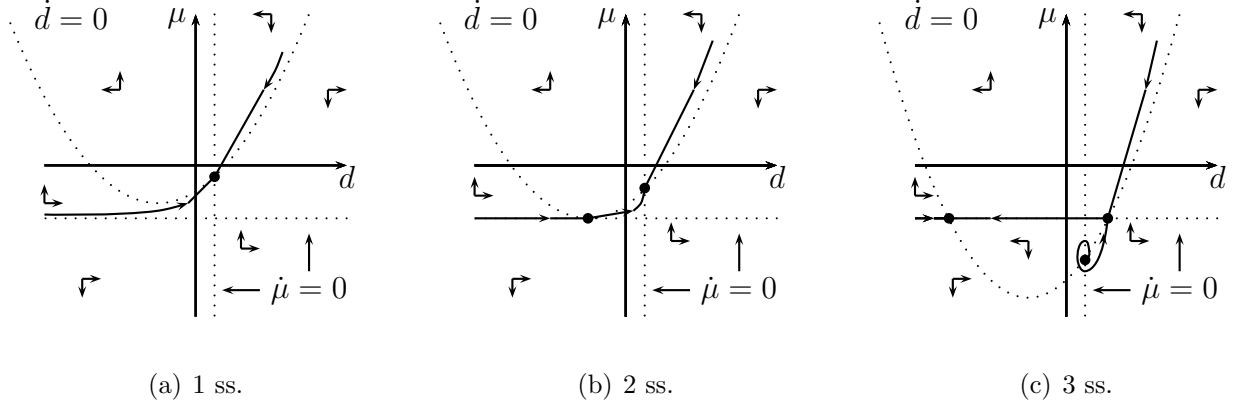


Figure 14: Phase plane diagram of system (5) if $\rho = 0$.

Additional Phase plane diagrams system (5).

In Figures 11 and 12 we visualized the phase plane diagrams of system (5) in case $\rho = 0$ and $\rho > 0$, respectively. Figure 11.c suggests that in case there are three steady states probably an unstable focus occurs at the steady state $d^e = \frac{\theta - \bar{r}}{2\alpha}$. This is confirmed, e.g., if we consider the case $\theta = 0.1$, $\bar{r} = 0.03$, $\alpha = 0.05$, $\beta_F = \beta_M = 0.04$, $\bar{d}_F = \bar{d}_M = 0.7$, $\bar{f} = 0$ and $\bar{m} = 1$. To prove this property for the general case seems, however, not to be completely trivial. A similar remark applies w.r.t. Figure 12.c. This figure suggests that the unstable steady state will always be a node. However, again, we could not directly come up with a conclusive argument here.

The Cooperative Case

Proof of Theorem 4.1.

The Hamiltonian for problem (12-13) is

$$H = \omega L_F + (1 - \omega)L_M + \lambda(\bar{r}d + \alpha d^2 + f - m).$$

The necessary conditions for existence of an optimal solution are then

$$\dot{d}(t) = \bar{r}d(t) + \alpha d^2(t) + f(t) - m(t) \quad (32)$$

$$\dot{\lambda}(t) = -\omega\beta_F e^{-\theta t}(d(t) - \bar{d}_F) - (1 - \omega)\beta_M e^{-\theta t}(d(t) - \bar{d}_M) - \lambda(t)(\bar{r} + 2\alpha d(t)) \quad (33)$$

$$e^{\theta t}\lambda(t) = -\omega(f(t) - \bar{f}) \quad (34)$$

$$e^{\theta t}\lambda(t) = (1 - \omega)(m(t) - \bar{m}). \quad (35)$$

From (34) and (35) we immediately have for any Pareto solution the next relationship between optimal monetary financing and fiscal deficits

$$f(t) - \bar{f} = -\frac{1 - \omega}{\omega}(m(t) - \bar{m}). \quad (36)$$

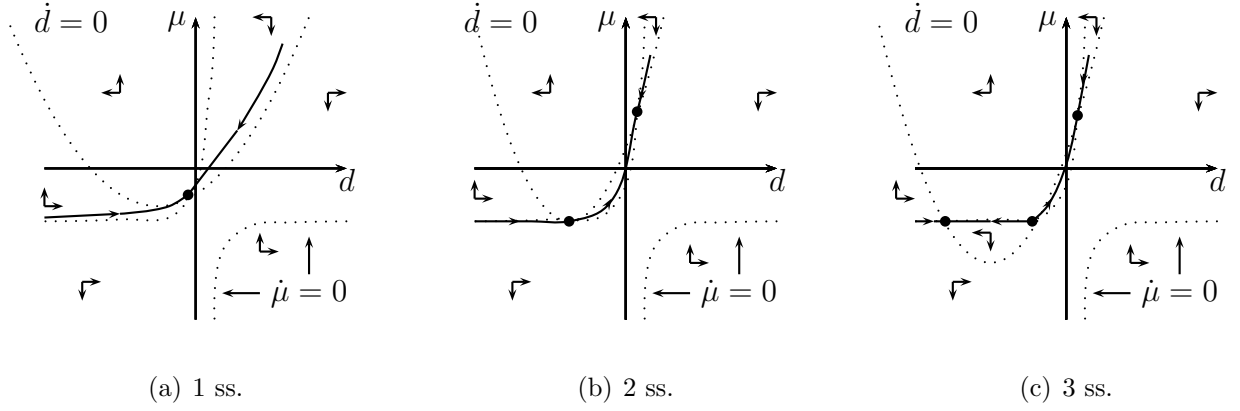


Figure 15: Phase plane diagram of system (5) if $\rho > 0$.

Introducing $\mu(t) := e^{\theta t} \lambda(t)$ the above equations (32-35) can be rewritten as

$$\begin{aligned} \dot{d}(t) &= \bar{r}d(t) + \alpha d^2(t) + f(t) - m(t) \\ \dot{\mu}(t) &= -\omega\beta_F(d(t) - \bar{d}_F) - (1 - \omega)\beta_M(d(t) - \bar{d}_M) - \mu(-\theta + \bar{r} + 2\alpha d(t)) \\ \mu(t) &= -\omega(f(t) - \bar{f}) \end{aligned} \quad (37)$$

$$\mu(t) = (1 - \omega)(m(t) - \bar{m}). \quad (38)$$

Since we are only interested in strategies which converge, it follows from (34) that μ should converge. So, the optimal control and state trajectory are solutions that converge to the steady states of system (39).

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{bmatrix} \alpha d^2(t) - \frac{1}{\omega(1-\omega)}\mu(t) + \bar{r}d(t) + \bar{f} - \bar{m} \\ (\theta - \bar{r} - 2\alpha d(t))\mu(t) + \omega\beta_F(\bar{d}_F - d(t)) + (1 - \omega)\beta_M(\bar{d}_M - d(t)) \end{bmatrix} =: \begin{bmatrix} f_{1,c}(d, \mu) \\ f_{2,c}(d, \mu) \end{bmatrix}. \quad (39)$$

Notice that $\frac{\partial f_{1,c}}{\partial d} + \frac{\partial f_{2,c}}{\partial \mu} = \theta$. So, if $\theta \neq 0$, by Bendixson's theorem, this system of differential equations has no periodic solutions. Furthermore the steady state values are obtained as the solutions d_c^e of

$$g_c(d) := \omega(1 - \omega)(\alpha d^2 + \bar{r}d + \bar{f} - \bar{m})(\theta - \bar{r} - 2\alpha d) + \omega\beta_F(\bar{d}_F - d) + (1 - \omega)\beta_M(\bar{d}_M - d) = 0, \quad (40)$$

with $\mu_c^e := \omega(1 - \omega)\{\bar{r}d_c^e + \alpha d_c^{e^2} + \bar{f} - \bar{m}\}$. Some rewriting of (40) shows that d_c^e is the solution of the third order polynomial equation (16). Moreover notice that (39) can be rewritten as (14). Finally notice that, if $d_c^e \geq 0$, $m_c^e - f_c^e = \bar{r}d_c^e + \alpha d_c^{e^2} \geq 0$. \square

Corollary 6.2 By (37), $f(t) = \bar{f} - \frac{1}{\omega}\mu(t)$, and by (38), $m(t) = \bar{m} + \frac{1}{1-\omega}\mu(t)$. Consequently, the steady state values are $f_c^e := \bar{f} - \frac{1}{\omega}\mu_c^e = \omega\bar{f} + (1 - \omega)\bar{m} - (1 - \omega)(\bar{r}d_c^e + \alpha d_c^{e^2})$ and $m_c^e := \bar{m} + \frac{1}{1-\omega}\mu_c^e = \omega\bar{f} + (1 - \omega)\bar{m} + \omega(\bar{r}d_c^e + \alpha d_c^{e^2})$, respectively.

Furthermore, by (36), for all t (and in particular for the converged variables), $\frac{\partial f(t)}{\partial \alpha} = -\frac{1-\omega}{\omega} \frac{\partial m(t)}{\partial \alpha}$ and $\frac{\partial f(t)}{\partial \theta} = -\frac{1-\omega}{\omega} \frac{\partial m(t)}{\partial \theta}$. \square

Lemma 6.3 *let $\omega \in (0, 1)$ be an arbitrary number and $\omega_1 := \sqrt{\omega(1-\omega)}$. $(d^*(.), \mu^*(.)) \in \mathcal{U}$ satisfy (5) if and only if $(d_c^*(.), \mu_c^*(.)) := (d^*(.), \omega_1^2 \mu^*(.)) \in \mathcal{U}$ satisfy (14) with β_F replaced by $(1-\omega)\beta_F$ and β_M replaced by $\omega\beta_M$.*

Proof.

Assume $(d^*(.), \mu^*(.)) \in \mathcal{U}$ solve (5). Next consider, for a fixed $\omega \in (0, 1)$, $(d_c^*(.), \mu_c^*(.)) := (d^*(.), \omega_1^2 \mu^*(.))$. Elementary calculations show then, that $(d_c^*(.), \mu_c^*(.))$ satisfy the set of differential equations

$$\begin{bmatrix} \dot{d}(t) \\ \dot{\mu}(t) \end{bmatrix} = \begin{bmatrix} \bar{r} & -\frac{1}{\omega(1-\omega)} \\ -\omega\tilde{\beta}_F - (1-\omega)\tilde{\beta}_M & \theta - \bar{r} \end{bmatrix} \begin{bmatrix} d(t) \\ \mu(t) \end{bmatrix} + \alpha d(t) \begin{bmatrix} d(t) \\ -2\mu(t) \end{bmatrix} + \begin{bmatrix} \bar{f} - \bar{m} \\ \omega\tilde{\beta}_F \bar{d}_F + (1-\omega)\tilde{\beta}_M \bar{d}_M \end{bmatrix},$$

where $\tilde{\beta}_F := (1-\omega)\beta_F$ and $\tilde{\beta}_M := \omega\beta_M$. That is, $(d_c^*(.), \mu_c^*(.))$ satisfies (14) with β_i replaced by $\tilde{\beta}_i$, $i = F, M$. \square

Proof Theorem 4.3.

Let $\tilde{d} := e^{-\frac{1}{2}\theta t}d(t)$; $c(t) := e^{-\frac{1}{2}\theta t}$; $\tilde{f}(t) := e^{-\frac{1}{2}\theta t}f(t)$; $\tilde{m}(t) := e^{-\frac{1}{2}\theta t}m(t)$; $x(t) := [\tilde{d}(t) \ c(t)]^T$ and $u(t) := [\tilde{f}(t) \ \tilde{m}(t)]^T$. Then problem (12-13) is equivalent to the minimization of

$$\frac{1}{2} \int_0^\infty [x^T(t) \ u^T(t)](\omega M_F + (1-\omega)M_M)[x^T(t) \ u^T(t)]^T dt \text{ s.t. } \dot{x}(t) = Ax(t) + Bu(t), \quad (41)$$

where

$$\begin{aligned} A &= \begin{bmatrix} \bar{r} - \frac{1}{2}\theta & 0 \\ 0 & -\frac{1}{2}\theta \end{bmatrix}; \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}; \quad x(0) = \begin{bmatrix} d_0 \\ 1 \end{bmatrix}; \quad M_i = \begin{bmatrix} Q_i & V_i \\ V_i^T & R_i \end{bmatrix}, \quad i = F, M; \\ Q_F &= \begin{bmatrix} \beta_F & -\beta_F \bar{d}_F \\ -\beta_F \bar{d}_F & \beta_F \bar{d}_F^2 + \bar{f}^2 \end{bmatrix}; \quad V_F = \begin{bmatrix} 0 & 0 \\ \bar{f} & 0 \end{bmatrix}; \quad R_F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad R_M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \\ Q_M &= \begin{bmatrix} \beta_M & -\beta_M \bar{d}_M \\ -\beta_M \bar{d}_M & \beta_M \bar{d}_M^2 + \bar{m}^2 \end{bmatrix}; \quad V_M = \begin{bmatrix} 0 & 0 \\ 0 & \bar{m} \end{bmatrix}; \quad M = \omega M_F + (1-\omega)M_M = \begin{bmatrix} Q & V \\ V^T & R \end{bmatrix}; \\ Q &= \begin{bmatrix} \omega\beta_F + (1-\omega)\beta_M & -(\omega\beta_F \bar{d}_F + (1-\omega)\beta_M \bar{d}_M) \\ -(\omega\beta_F \bar{d}_F + (1-\omega)\beta_M \bar{d}_M) & \omega(\beta_F \bar{d}_F^2 + \bar{f}^2) + (1-\omega)(\beta_M \bar{d}_M^2 + \bar{m}^2) \end{bmatrix}; \\ V &= \begin{bmatrix} 0 & 0 \\ \omega\bar{f} & (1-\omega)\bar{m} \end{bmatrix} \text{ and } R = \begin{bmatrix} \omega & 0 \\ 0 & (1-\omega) \end{bmatrix}. \end{aligned}$$

The solution of this optimization problem (41) is (see e.g. [10])

$$u^*(t) = -R^{-1}(B^T K^+ + V)x(t),$$

where K^+ is the unique stabilizing solution of the algebraic Riccati equation

$$A^T K + KA - (KB + V)R^{-1}(B^T K + V^T) + Q. \quad (42)$$

Furthermore the minimal cost of (41) are $x_0^T K^+ x_0$. The corresponding cost using the optimal control for the individual players are, with $A_{cl} := A - BR^{-1}(B^T K^+ + V)$, $L_i = \frac{1}{2}x_0^T \tilde{M}_i x_0$, $i = F, M$, where \tilde{M}_i is the unique solution of the Lyapunov equation

$$A_{cl}^T \tilde{M}_i + \tilde{M}_i A_{cl} = -[I - (K^+ B + V)R^{-1}]M_i[I - (K^+ B + V)R^{-1}]^T.$$

From this the conclusions stated in the theorem readily follow. \square

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